MATH301
quiz #2, 04/05/18
Total Possible 100
Solutions

By enrolling in this course, each student assumes the responsibilities of an active participant in UMBC's scholarly community in which everyone's academic work and behavior are held to the highest standards of honesty. Cheating, fabrication, plagiarism, and helping others to commit these acts are all forms of academic dishonesty, and they are wrong. Academic misconduct could result in disciplinary action that may include, but is not limited to, suspension or dismissal.

Show all work legibly. **Name**:______________________

1. (40)

   (a) (10) Show that $\forall$ real $a \geq 0$ one has $(1+a)^n \geq 1 + na + \frac{n(n-1)}{2}a^2$, $n = 1, 2, \ldots$.

   Solution. Use induction. The statement is correct for $k = 1$. Assuming the statement is correct for $k = n$ one has

   $$(1+a)^{(n+1)} \geq (1+a)\left(1 + na + \frac{n(n-1)}{2}a^2\right) > 1 + (n+1)a + \frac{(n+1)n}{2}a^2.$$

   (b) (10) Prove that $\forall$ real $a > 0$ $\exists N$ such that $(n-1)\frac{a^2}{2} > 1$ when $n \geq N$.

   Solution. Pick any $N > \frac{2}{a^2} + 1$.

   (c) (20) Find $\lim_{n\to\infty} n^{\frac{1}{n}}$.

   Solution. Let a be a positive number. Pick N as in 1(b), then for each $n > N$ one has

   $$(1+a)^n \geq 1 + na + \frac{n(n-1)}{2}a^2 > n, \text{ and } 1 + a > n^{\frac{1}{n}} > 1.$$

2. (20) Let $a = \frac{4}{9}$. Find a sequence of integers $\{a_1, a_2, \ldots\}$, $a_i \in \{0, 1, 2\}$ so that

   $$a = \frac{a_1}{3^1} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \ldots.$$

Solution. $a = \frac{4}{9} = \frac{1}{3^1} + \frac{1}{3^2} + \frac{0}{3^3} + \ldots$, i.e., $a_1 = 1$, $a_2 = 1$, $a_n = 0$, $n \geq 3$.

$a_1 =$
$a_2 =$
$a_3 =$
$\dots$
$a_n =$

3. (20) Let $I_n = [-n, n]$, and $J_n = \mathbf{R} \setminus I_n$ (the complement of I_n relative to $\mathbf{R}$). Describe $\bigcap_{n=1}^{\infty} J_n$.

   Solution. Let $x \in \mathbf{R}$. There is n so that $|x| < n$, and $x \notin J_n$, hence $x \notin \bigcap_{n=1}^{\infty} J_n$.

   $\bigcap_{n=1}^{\infty} J_n =$

4. (20) Let x_n be a monotonically increasing and bounded sequence of real numbers (i.e., there is a number M so that for each $n = 1, 2, \dots$ one has $x_n \le M$, and $x_n \le x_{n+1}$). Let $[x, M] = \bigcap_{n=1}^{\infty} [x_n, M]$. True or False? $\lim_{n\to\infty} x_n = x$.

   Solution. Clearly $x_n \le x$ for each $n = 1, 2, \dots$. Select $\epsilon > 0$, and consider the interval $[x - \epsilon, M]$. If for each n one has $x_n \le x - \epsilon$, then

   $$[x - \epsilon, M] \subseteq \bigcap_{n=1}^{\infty} [x_n, M] = [x, M].$$

   The contradiction shows that there is N such that $x - \epsilon < x_N \le x$, and $|x_N - x| < \epsilon$. Note that when $N \le n$ one has $x - \epsilon < x_N < x_n \le x$, and $|x_n - x| < \epsilon$.

   Mark one and explain.
   □ True □ False

5. (20) Let $\{x_{i1}, x_{i2}, \dots, x_{in}, \dots\}$ be a family of sequences $i = 1, 2, \dots$. Assume that for each i the sequence $\{x_{in}\}$ converges, that is $\lim_{n\to\infty} x_{in} = x_i$, $i = 1, 2, \dots$. Assume that $\lim_{i\to\infty} x_i = x$. True or False? The sequence $\{x_{nn}\}$ converges to x as $n \to \infty$.

   Solution. Let

   $$x_{in} = \begin{cases} 0 & \text{when} \quad n < i \\ (-1)^n & \text{when} \quad i = n \\ \frac{1}{n} & \text{when} \quad n > i \end{cases}$$

   Note that $x_{nn} = (-1)^n$, hence the sequence $\{x_{nn}\}$ diverges.

   Mark one and explain.
   □ True □ False