MATH301
quiz #3, 05/08/18
Total Possible 100
Solutions

By enrolling in this course, each student assumes the responsibilities of an active participant in UMBC's scholarly community in which everyone's academic work and behavior are held to the highest standards of honesty. Cheating, fabrication, plagiarism, and helping others to commit these acts are all forms of academic dishonesty, and they are wrong. Academic misconduct could result in disciplinary action that may include, but is not limited to, suspension or dismissal.

Show all work legibly. **Name**:______________________

1. (20) True or False? The sequence $x_k = \sum_{n=1}^{k} \frac{1}{(n+2)(n+3)}$ converges.

   Solution. Note that $\frac{1}{(n+2)(n+3)} = \left[\frac{1}{n+2} - \frac{1}{n+3}\right]$, and

$$\begin{aligned}\sum_{n=0}^{k} \frac{1}{(n+2)(n+3)} &= \left[\frac{1}{2+0} - \frac{1}{2+1}\right] \\ &+ \left[\frac{1}{2+1} - \frac{1}{2+2}\right] \\ &+ \dots \\ &+ \left[\frac{1}{2+k} - \frac{1}{2+k+1}\right] \\ &= \frac{1}{2} - \frac{1}{2+k+1}.\end{aligned}$$

   Mark one and explain.
   ▫ True ▫ False

2. (20) Let $\{x_n\}$ be a Cauchy sequence. True or False? If x_n is an integer for each $n = 1, 2, \dots$, then $\lim_{n\to\infty} x_n$ is an integer.

   Solution. Denote $\lim_{n\to\infty} x_n$ by x. Assume the opposite, i.e. $x \notin \mathbf{N}$. Let $\epsilon > 0$ so that $\epsilon < \inf\{|x-k| \ : \ k \in \mathbf{N}\}$. Select n such that $|x_n - x| < \epsilon$. This implies that $x_n \notin \mathbf{N}$. This contradiction completes the proof.

   Mark one and explain.
   ▫ True ▫ False

3. (20) Let $\{x_n\}$ be an unbounded sequence. True or False? There is a subsequence $\{x_{n_k}\}$ so that $\lim_{k\to\infty} \frac{1}{x_{n_k}} = 0$.

Solution. Let n_1 be an integer such that $|x_{n_1}| > 1$. Define n_k, $k = 2, 3, \ldots$ as an integer so that

$$n_k > n_{k-1} \text{ and } |x_{n_k}| > k.$$

Note that $\left|0 - \frac{1}{x_{n_k}}\right| = \left|\frac{1}{x_{n_k}}\right| < \frac{1}{k}$.

Mark one and explain.

▫ True ▫ False

4. (20) Let $x_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+n}$. True or False? The sequence $\{x_n\}$ converges.

Solution. Note that $0 < x_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+n} < \frac{n}{n+1} < 1$, and

$$x_{n+1} - x_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} > \frac{1}{2n+2} + \frac{1}{2n+2} - \frac{1}{n+1} = 0.$$

Mark one and explain.

▫ True ▫ False

5. (20) Let $0 < a < b$, and $x_n = \left(\frac{a^{n+1} + b^{n+1}}{a^n + b^n}\right)$. Compute $\lim_{n\to\infty} x_n$, if exists.

Solution. Note that

$$x_n = \left(\frac{a^{n+1} + b^{n+1}}{a^n + b^n}\right) = \left(\frac{a\left(\frac{a}{b}\right)^n + b}{\left(\frac{a}{b}\right)^n + 1}\right), \text{ hence } \lim_{n\to\infty} x_n = b.$$

$\lim_{n\to\infty} x_n =$

6. (20) Let $\{x_n\}$ be a sequence of positive real numbers so that $\lim_{n\to\infty} x_n^{\frac{1}{n}} = L < 1$. True or False? There exists $r \in (0, 1)$ and a positive integer N so that $x_n < r^n$ for each $n \geq N$.

Solution. If $r = \frac{1+L}{2}$, then $0 \leq L < r < 1$. For $\epsilon = r - L$ there is N so that for each $n \geq N$ one has $\left|x_n^{\frac{1}{n}} - L\right| < \epsilon$. Hence

$$x_n^{\frac{1}{n}} < L + \epsilon = r \text{ and } x_n < r^n.$$

Mark one and explain.

▫ True ▫ False