MATH221

December 12, 2019

final exam

total 200

Solutions

1. (20) Solve the system of linear equations

$$\begin{array}{rcrcrcr} x & - & y & & & = & 1 \\ 2x & + & y & - & 3z & = & 8 \\ x & - & 2y & + & 3z & = & -5 \end{array}$$

Solution.

$$\begin{bmatrix} 1 & -1 & 0 & 1 \\ 2 & 1 & -3 & 8 \\ 1 & -2 & 3 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 3 & -3 & 6 \\ 0 & -1 & 3 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 3 & -6 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$x = 1$, $y = 0$, $z = -2$.

2. (20) Consider the system of linear equations

$$\begin{array}{rcrcrcr} x & - & y & & & = & a \\ 2x & + & y & - & 3z & = & 8 \\ x & - & 2y & + & 3z & = & -5 \end{array}$$

If $x = 1$ find a.

Solution. The system becomes

$$\begin{array}{rcrcrcr} -a & - & y & & & = & -1 \\ & & y & - & 3z & = & 6 \\ & - & 2y & + & 3z & = & -6 \end{array} \quad \text{and } a = 1.$$

$a =$

3. (60) Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & -3 \\ 1 & -2 & 3 \end{bmatrix}$.

   (a) (15) Find Nul A.
   Nul $A = \{0\}$.

(b) (15) Find the column space Col A

Col $A = \mathbf{R}^3$

(c) (15) True or False? A^{-1} exists.

Mark one and explain.

□ True □ False

(d) (15) Compute det A.

det $A = 6$

4. (20) True or False? If the only solution for $A\mathbf{x} = 0$ is $\mathbf{x} = 0$, then for each $\mathbf{b}$ the system $A\mathbf{x} = \mathbf{b}$ has only one solution.

Solution. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

While $A\mathbf{x} = 0$ has only one solution $A\mathbf{x} = \mathbf{b}$ has no solutions at all.

Mark one and explain.

□ True □ False

5. (20) The three elementary row operations are:

(a) Interchage two rows.

(b) Replace one row by the sum of itself and a multiple of another row.

(c) Multiply all entries in a row by a nonzero constant.

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. True or False? A can be transformed to B using only two last elementary row operations.

Solution.

$$A = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \to \begin{bmatrix} R_1 + R_2 \\ R_2 \end{bmatrix} \to \begin{bmatrix} R_1 + R_2 \\ -R_1 \end{bmatrix} \to \begin{bmatrix} R_2 \\ -R_1 \end{bmatrix} \to \begin{bmatrix} R_2 \\ R_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Mark one and explain.

□ True □ False

6. (40) Let $A = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$.

(a) (15) Find eigenvalues λ_1 and λ_2.

Solution. $\lambda_1 = 0$, $\lambda_2 = 5$

$\lambda_1 =$ $\lambda_2 =$

(b) (15) Find eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

Solution. A possible choice of eigenvectors is $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

$\mathbf{v}_1 =$ $\mathbf{v}_2 =$

(c) (10) Find a diagonal matrix D, and an invertible matrix V, and V^{-1} so that $A = VDV^{-1}$.

Solution.

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}, V = \begin{bmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}, \; V^{-1} = V.$$

$D =$ $, V =$ $, V^{-1} =$

7. (20) Let $\mathbf{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Find the orthogonal projection $\mathbf{p}$ of the vector $\mathbf{v}$ on the line $\{t\mathbf{a} + \mathbf{b}, \; -\infty < t < \infty\}$.

Solution. Let $\mathbf{p} = t\mathbf{a} + \mathbf{b}$, then $0 = \mathbf{a}^T(\mathbf{v} - \mathbf{p})$, and $t = -\dfrac{1}{2}$. Finally $\mathbf{p} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$.

$\mathbf{p} =$