MATH603
quiz 0
09/03/19
Total 100
Solutions

By enrolling in this course, each student assumes the responsibilities of an active participant in UMBC's scholarly community in which everyone's academic work and behavior are held to the highest standards of honesty. Cheating, fabrication, plagiarism, and helping others to commit these acts are all forms of academic dishonesty, and they are wrong. Academic misconduct could result in disciplinary action that may include, but is not limited to, suspension or dismissal.

Show all work legibly. **Name**:______________________

1. (30) Let $\mathbf{u}_1,\ \mathbf{u}_2$ be linearly independent vectors of magnitude 1 (i.e., $\mathbf{u}_1^T\mathbf{u}_1 = \mathbf{u}_2^T\mathbf{u}_2 = 1$).

   (a) (10) True or False? $\left|\mathbf{u}_1^T\mathbf{u}_2\right| \le 1$

   Solution. Note that for each real number t

   $$0 \le (\mathbf{u}_1 - t\mathbf{u}_2)^T(\mathbf{u}_1 - t\mathbf{u}_2) = t^2\mathbf{u}_2^T\mathbf{u}_2 - 2t\mathbf{u}_2^T\mathbf{u}_1 + \mathbf{u}_1^T\mathbf{u}_1 = t^2 - t\left(2\mathbf{u}_2^T\mathbf{u}_1\right) + 1.$$

   This yields

   $$\left(2\mathbf{u}_2^T\mathbf{u}_1\right)^2 - 4 \le 0, \text{ and } \left|\mathbf{u}_1^T\mathbf{u}_2\right| \le 1.$$

   Mark one and explain.
   ▫ True ▫ False

   (b) (10) True or False? If $\left|\mathbf{u}_1^T\mathbf{u}_2\right| = 1$, then $\mathbf{u}_1 = \pm\mathbf{u}_2$

   Solution. When $\mathbf{u}_1^T\mathbf{u}_2 = 1$ one has $|\mathbf{u}_1 - t\mathbf{u}_2|^2 = (t-1)^2$. When $t = 1$ one has $|\mathbf{u}_1 - \mathbf{u}_2|^2 = 0$, and $\mathbf{u}_1 = \mathbf{u}_2$. If $\mathbf{u}_1^T\mathbf{u}_2 = -1$, one has $|\mathbf{u}_1 - t\mathbf{u}_2|^2 = (t+1)^2$. For $t = -1$ this yields $|\mathbf{u}_1 + \mathbf{u}_2|^2 = 0$, and $\mathbf{u}_1 = -\mathbf{u}_2$. Note that the conditions

   i. $\mathbf{u}_1,\ \mathbf{u}_2$ are linearly independent vectors of magnitude 1,
   ii. $\left|\mathbf{u}_1^T\mathbf{u}_2\right| = 1$

   are mutually exclusive.

   Mark one and explain.
   ▫ True ▫ False

(c) (10) Let $\mathbf{v}_1,\ \mathbf{v}_2$ be linearly independent vectors of magnitude 1. True or False? If $\mathbf{u}_i^T\mathbf{v}_j = 0$ for each i, j then the set $\{\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{v}_1,\ \mathbf{v}_2\}$ is linearly independent.

Solution. Consider the equation

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + d_1\mathbf{v}_1 + d_2\mathbf{v}_2 = 0.$$

The dot product of $\mathbf{u}_1$ with this equation yields $c_1 + c_2\mathbf{u}_1^T\mathbf{u}_2 = 0$. The dot product of $\mathbf{u}_2$ with this equation yields $c_1\mathbf{u}_1^T\mathbf{u}_2 + c_2 = 0$. The two equations lead to

$$c_1\left(1 - \left|\mathbf{u}_1^T\mathbf{u}_2\right|^2\right) = 0.$$

Since $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly independent vectors $1 - \left|\mathbf{u}_1^T\mathbf{u}_2\right|^2 > 0$ (see (a) and (b) above), hence $c_1 = 0$. By the same token $c_2 = d_1 = d_2 = 0$.

Mark one and explain.

▫ True ▫ False

2. (20) Let $\mathbf{v}_1,\ \mathbf{v}_2$ and $\mathbf{u}_1,\ \mathbf{u}_2$ be two pairs of linearly independent vectors. True or False? $\mathbf{v}_1\mathbf{u}_1^T + \mathbf{v}_2\mathbf{u}_2^T$ has rank 2.

Solution. First note that $\text{rank}AB \le \text{rank}B$ for any pair of matrices A and B when the product AB is defined. If $U = [\mathbf{u}_1, \mathbf{u}_2]$, and $V = [\mathbf{v}_1, \mathbf{v}_2]$, then $\mathbf{v}_1\mathbf{u}_1^T + \mathbf{v}_2\mathbf{u}_2^T = VU^T$, this shows that $\text{rank}\left(\mathbf{v}_1\mathbf{u}_1^T + \mathbf{v}_2\mathbf{u}_2^T\right) \le \text{rank}U^T = 2$.

Since $\mathbf{u}_1,\ \mathbf{u}_2$ are linearly independent vectors at least one coordinate of $\mathbf{u}_1$ is different from 0. If $\mathbf{u}_{1i} \ne 0$, then the i^{th} column of $\mathbf{v}_1\mathbf{u}_1^T + \mathbf{v}_2\mathbf{u}_2^T$, vector $u_{1i}\mathbf{v}_1 + u_{2i}\mathbf{v}_2 \ne 0$. Hence $\text{rank}\left(\mathbf{v}_1\mathbf{u}_1^T + \mathbf{v}_2\mathbf{u}_2^T\right) \ge 1$.

We next show that $\text{rank}\left(\mathbf{v}_1\mathbf{u}_1^T + \mathbf{v}_2\mathbf{u}_2^T\right) \ge 2$. Suppose the opposite, i.e.

$$\text{rank}\left(\mathbf{v}_1\mathbf{u}_1^T + \mathbf{v}_2\mathbf{u}_2^T\right) = 1.$$

Then for each pair $i \ne j$ the vectors

$$u_{1i}\mathbf{v}_1 + u_{2i}\mathbf{v}_2 \text{ and } u_{1j}\mathbf{v}_1 + u_{2j}\mathbf{v}_2$$

are linearly dependent, that is there are scalars c_1 and c_2, not all 0 so that

$$c_1\left(u_{1i}\mathbf{v}_1 + u_{2i}\mathbf{v}_2\right) + c_2\left(u_{1j}\mathbf{v}_1 + u_{2j}\mathbf{v}_2\right) = 0.$$

In other words

$$\left(c_1u_{1i} + c_2u_{1j}\right)\mathbf{v}_1 + \left(c_1u_{2i} + c_2u_{2j}\right)\mathbf{v}_2 = 0,$$

and

$$\det \begin{bmatrix} u_{1i} & u_{1j} \\ u_{2i} & u_{2j} \end{bmatrix} = 0.$$

Since the det is 0 for each pair of indecies i, j the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly dependent. This contradiction completes the proof.

Mark one and explain.
□ True □ False

3. (20) Let A be an $n \times n$ matrix so that $A^T A = I$. True or False? $\det A^2 = 1$.

Solution. Since $\det A^T = \det A$ one has $\det A^2 = \det A^T A = \det I = 1$.

Mark one and explain.
□ True □ False

4. (20) Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. If $\lambda_1 = 2$ and $\lambda_2 = 3$ are the eigenvalues of A compute $a_{11} + a_{22}$, and $\det A$.

Solution. $\operatorname{tr} A = a_{11} + a_{22} = \lambda_1 + \lambda_2 = 5$, $\det A = \lambda_1 \lambda_2 = 6$.

$a_{11} + a_{22} =$
$\det A =$

5. (20) Let $\mathbf{v}$, $\mathbf{w} \in \mathbf{R}^n$, and $a = \mathbf{v}^T \mathbf{w}$. Consider an $n \times n$ matrix $A = \mathbf{v}\mathbf{w}^T$.

   (a) (10) Show that a and 0 are eigenvalues of A. Find an eigenvector $\mathbf{u}$ that corresponds to the eigenvalue a.

   Solution. If $\mathbf{u} = \mathbf{v}$, then $A\mathbf{u} = A\mathbf{v} = \mathbf{v}\mathbf{w}^T\mathbf{v} = a\mathbf{v} = a\mathbf{u}$.

   (b) (20) Find dimension $\dim V_a$ of the eigenspace that corresponds to the eigenvalue a, and dimension $\dim V_0$ of the eigenspace that corresponds to the eigenvalue 0.

   Solution.
   Case 1 Note that when $\mathbf{w} = 0$ the spaces V_a and V_0 are identical, and $V_a = V_0 = \mathbf{R}^n$.
   Case 2 We now assume that $\mathbf{w} \neq 0$. If $a = \mathbf{w}^T\mathbf{v} = 0$, then $V_a = V_0$. If $\mathbf{v} = 0$, then $V_a = V_0 = \mathbf{R}^n$. If $\mathbf{v} \neq 0$, then an eigenvector $\mathbf{u}$ should satisfy $0\mathbf{u} = A\mathbf{u} = \mathbf{v}(\mathbf{w}^T\mathbf{u})$, that is $\mathbf{u} \in \left\{\mathbf{w}^\perp\right\}$, and $\dim V_0 = \dim V_a = n - 1$.
   Case 3 Finally we focus on the case $\mathbf{w} \neq 0$, and $a = \mathbf{w}^T\mathbf{v} \neq 0$. If $\mathbf{u}$ is an eigenvector of A with the eigenvalue $a = \mathbf{w}^T\mathbf{v}$, then $0 \neq a\mathbf{u} = A\mathbf{u} = \mathbf{v}(\mathbf{w}^T\mathbf{u})$. This shows that $\mathbf{u} \in \operatorname{span}\{\mathbf{v}\}$, and $\dim V_a = 1$. On the other hand when $0 = A\mathbf{u} = \mathbf{v}(\mathbf{w}^T\mathbf{u})$ the dot product $\mathbf{w}^T\mathbf{u} = 0$, $\mathbf{u} \in \left\{\mathbf{w}^\perp\right\}$, and $\dim V_0 = n - 1$.