MATH603
test 2
12/03/2019
Total 100
Solutions

By enrolling in this course, each student assumes the responsibilities of an active participant in UMBC's scholarly community in which everyone's academic work and behavior are held to the highest standards of honesty. Cheating, fabrication, plagiarism, and helping others to commit these acts are all forms of academic dishonesty, and they are wrong. Academic misconduct could result in disciplinary action that may include, but is not limited to, suspension or dismissal.

Show all work legibly. **Name**:______________________

1. (40) Let $f(\mathbf{x}) = \sum_{i=1}^{n} x_i^2$, and $g(\mathbf{x}) = \sum_{i=1}^{n} |x_i|$.

   (a) (20) True or False? f is continious at 0 with respect to g.

   Solution. Let ϵ be a positive number. If $0 < \delta < \min\{\epsilon, 1\}$, and $g(\mathbf{x}) < \delta$, then $f(\mathbf{x}) < g(\mathbf{x}) < \delta < \epsilon$.
   Mark one and explain.
   □ True □ False

   (b) (20) True or False? g is continious at 0 with respect to f.

   Solution. Let ϵ be a positive number. If $0 < \delta < \frac{\epsilon^2}{n^2}$, and $f(\mathbf{x}) = \sum_{i=1}^{n} x_i^2 < \delta$, then $x_i^2 < \delta$, and $|x_i| < \sqrt{\delta}$, $i = 1, \dots, n$. This yields $g(\mathbf{x}) = \sum_{i=1}^{n} |x_i| < n\sqrt{\delta} < \epsilon$.
   Mark one and explain.
   □ True □ False

2. (20) Let $f(\mathbf{x}) = \left(|x_1|^{\frac{1}{2}} + \dots + |x_n|^{\frac{1}{2}}\right)^2$. True or False? $f(\mathbf{x})$ is a norm.

   Solution. Note that $f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = (1+1)^2 > 1^2 + 1^2 = f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + f\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$.
   Mark one and explain.
   □ True □ False

3. (40) Consider $m \times n$ orthonormal matrices Q_1 and Q_2, and an upper diagonal $n \times n$ matrix R with positive diagonal elements.

(a) (20) True or False? If $Q_1 = Q_2 R$, then $R = I$, and $Q_1 = Q_2$.

Solution. Let $Q_1 = [\mathbf{q}'_1, \ldots, \mathbf{q}'_n]$ and $Q_2 = [\mathbf{q}''_1, \ldots, \mathbf{q}''_n]$. Since $\mathbf{q}'_1 = r_{11}\mathbf{q}''_1$ one has $r_{11} = 1$, and $\mathbf{q}'_1 = \mathbf{q}''_1 = \mathbf{q}_1$. Next consider the second columns of Q_1 and Q_2. One has

$$\mathbf{q}'_2 = r_{12}\mathbf{q}_1 + r_{22}\mathbf{q}''_2.$$

Dot multiplication of both sides of the equation by $\mathbf{q}_1$ leads to $r_{12} = 0$, and this yields $r_{22} = 1$, and $\mathbf{q}'_2 = \mathbf{q}''_2 = \mathbf{q}_2$.

Next assume that $Q_1 = [\mathbf{q}_1, \ldots, \mathbf{q}_k, \mathbf{q}'_{k+1}, \ldots, \mathbf{q}'_n]$, $Q_2 = [\mathbf{q}_1, \ldots, \mathbf{q}_k, \mathbf{q}''_{k+1}, \ldots, \mathbf{q}''_n]$, and

$$R = \left[\begin{array}{cccccc} \mathbf{e}_1 & \ldots & \mathbf{e}_k & \mathbf{r}_{k+1} & \ldots & \mathbf{r}_n \end{array}\right] \text{ with } \mathbf{r}_{k+1} = \left[\begin{array}{l} r_{1,k+1} \\ r_{1,k+1} \\ \ldots \\ r_{k+1,k+1} \\ 0 \\ \ldots \\ 0 \end{array}\right].$$

Note that

$$\mathbf{q}'_{k+1} = r_{1,k+1}\mathbf{q}_1 + \ldots + r_{k,k+1}\mathbf{q}_k + r_{k+1,k+1}\mathbf{q}''_{k+1}.$$

Dot multiplication of both sides of the equation by $\mathbf{q}_i$, $i = 1, \ldots, k$ yields

$$r_{1,k+1} = \ldots = r_{k,k+1} = 0.$$

This in turn leads to $r_{k+1,k+1} = 1$, and $\mathbf{q}'_{k+1} = \mathbf{q}'_{k+1} = \mathbf{q}_{k+1}$. Repetition of this argument yields the result.

Mark one and explain.

□ True □ False

(b) (20) True or False? If R_1 and R_2 upper diagonal $n \times n$ matrices with positive diagonal elements such that $Q_1 R_1 = Q_2 R_2$, then $Q_1 = Q_2$, and $R_1 = R_2$.

Solution. Note that $Q_1 R_1 = Q_2 R_2$ yields $Q_1 = Q_2 \left(R_2 R_1^{-1}\right)$.

Mark one and explain.

□ True □ False