MATH225
quiz #4, 05/07/19
Total 100
Solutions

Show all work legibly. **Name**:______________________

1. (20) Solve the initial value problem $y''(t) + 4y'(t) + 13y(t) = 3\delta(t)$, $y(0) = y'(0) = 0$.

   $y(t) =$

   Solution. Applying transform to both sides one gets

   $$(s^2 + 4s + 13)Y(s) = 3, \text{ and } Y(s) = \frac{3}{s^2 + 4s + 13} = \frac{3}{(s+2)^2 + 3^2}.$$

   Finally $y(t) = L^{-1}\left(\frac{3}{(s+2)^2 + 3^2}\right) = e^{-2t}\sin 3t$.

2. (40)

   (a) (20) Find the inverse Laplace Transform $L^{-1}\left(e^{-4s}\frac{1}{s(s+2)}\right)$.

   $y(t) =$

   Solution. Note that

   $$\frac{1}{s(s+2)} = \frac{a}{s} + \frac{b}{s+2}, \text{ i.e. } a = \frac{1}{2}, \text{ and } b = -\frac{1}{2}, \text{ that is } \frac{1}{s(s+2)} = \frac{1}{2}\left[\frac{1}{s} - \frac{1}{s+2}\right].$$

   $$\begin{aligned} L^{-1}\left(e^{-4s}\frac{1}{s(s+2)}\right) &= \frac{1}{2}\left[L^{-1}\left(e^{-4s}\frac{1}{s}\right) - L^{-1}\left(e^{-4s}\frac{1}{(s+2)}\right)\right] \\ &= \frac{1}{2}u(t-4)\left[1 - e^{-2(t-4)}\right]. \end{aligned}$$

   (b) (20) Solve the initial value problem $y'(t) + 2y = u(t-4)$, $y(0) = 3$.

   $y(t) =$

   Solution. Applying transform to both sides one gets

   $$Y(s) = 3L\left(e^{-2t}\right) + e^{-4s}\frac{1}{s(s+2)}.$$

   and

   $$y(t) = L^{-1}\left(3L\left(e^{-2t}\right) + e^{-4s}\frac{1}{s(s+2)}\right) = 3e^{-2t} + \frac{1}{2}u(t-4)\left[1 - e^{-2(t-4)}\right].$$

3. (20) Find the Laplace Transform $Y(s)$ of $y(t) = \begin{cases} 0 & t < 1 \\ t^2 - 2t + 2 & t \geq 1 \end{cases}$

$Y(s) =$

Solution. Note that $y(t) = u(t-1)(t-1)^2 + u(t-1)$. Hence $Y(t) = e^{-t}\dfrac{2}{s^3} + \dfrac{e^{-s}}{s}$.

4. (20) Let $f(t)$ be a function defined for $0 \leq t \leq 2$ by

$$f(t) = \begin{cases} t & \text{when} \quad 0 \leq t \leq 1 \\ t-1 & \text{when} \quad 1 \leq t \leq 2 \end{cases} \quad \text{and } f(t+2) = f(t).$$

Find $F(s)$ the Laplace Transform of $f(t)$. (Hint: $\displaystyle\int xe^{ax}\,dx = \left(\frac{x}{a} - \frac{1}{a^2}\right)e^{ax}$)

$F(s) =$

Solution.

$$\begin{aligned} \int_0^1 e^{-st} f(t)\,dt &= \int_0^1 te^{-st}\,dt = \left(\frac{t}{-s} - \frac{1}{s^2}\right)e^{-ts}\Big|_0^1 \\ &= \frac{1}{s^2} - \left(\frac{1}{s} + \frac{1}{s^2}\right)e^{-s} = \frac{1}{s^2} - e^{-s}\frac{s+1}{s^2} \\ &= \frac{1 - e^{-s} - se^{-s}}{s^2} = \frac{(1-e^{-s}) + s(1-se^{-s}) - s}{s^2}. \end{aligned}$$

Since f is a periodic function with period 1 one has

$$F(s) = F_T(s)\frac{1}{1-e^{-s}} = \frac{1}{s^2} + \frac{1}{s} - \frac{1}{s}\frac{1}{1-e^{-s}}.$$