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Claim:

Let $\varphi(n)$ denote Euler's totient function. Then, for any n with prime factorization $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}$

we have that

$$\varphi(n) = n$$

$$- \left(p_1^{\alpha_1-1} p_2^{\alpha_2} \dots p_m^{\alpha_m} + p_1^{\alpha_1} p_2^{\alpha_2-1} p_3^{\alpha_3} \dots p_m^{\alpha_m} + \dots + p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m-1}\right)$$

$$+(p_1^{\alpha_1-1} p_2^{\alpha_2-1} p_3^{\alpha_3} \dots p_m^{\alpha_m} + p_1^{\alpha_1-1} p_2^{\alpha_2} p_3^{\alpha_3-1} p_4^{\alpha_4} \dots p_m^{\alpha_m} + \dots + p_1^{\alpha_1-1} p_2^{\alpha_2} p_3^{\alpha_3} \dots p_m^{\alpha_{m-1}}$$
$$+ p_1^{\alpha_1} p_2^{\alpha_2-1} p_3^{\alpha_3-1} p_4^{\alpha_4} \dots p_m^{\alpha_m} + p_1^{\alpha_1} p_2^{\alpha_2-1} p_3^{\alpha_3} p_4^{\alpha_4-1} p_5^{\alpha_5} \dots p_m^{\alpha_m} + \dots + p_1^{\alpha_1} p_2^{\alpha_2-1} p_3^{\alpha_3} \dots p_m^{\alpha_m-1}$$
$$+ \dots + p_1^{\alpha_1} p_2^{\alpha_2} \dots p_{m-1}^{\alpha_{m-1}-1} p_m^{\alpha_m-1})$$

$$-$$

$$\vdots$$

$$+(-1)^m \left(p_1^{\alpha_1-1} p_2^{\alpha_2-1} \dots p_m^{\alpha_m-1}\right)$$

Which can also be written as

$$\varphi(n) = n + \sum_{j=1}^{m} (-1)^j \left(\sum_{k=0}^{j-1} \left(\sum_{i_{j,k}=1}^{m-k} \sum_{i_{j,k-1}=\ i_{j,k}+1}^{m-(k-1)} \dots \sum_{i_{j,1}\ =\ i_{j,2}+1}^{m-1} \sum_{i_{j,0}=i_{j,1}+1}^{m} n/(p_{i_{j,0}} p_{i_{j,1}} \dots p_{i_{j,k}}) \right) \right)$$

Proof:

First, we begin with expression (A)

expression (A):

$$\left(p_1^{\alpha_1-1} p_2^{\alpha_2} \dots p_m^{\alpha_m} + p_1^{\alpha_1} p_2^{\alpha_2-1} p_3^{\alpha_3} \dots p_m^{\alpha_m} + \dots + p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m-1}\right)$$

$$-(p_1^{\alpha_1-1} p_2^{\alpha_2-1} p_3^{\alpha_3} \dots p_m^{\alpha_m} + p_1^{\alpha_1-1} p_2^{\alpha_2} p_3^{\alpha_3-1} p_4^{\alpha_4} \dots p_m^{\alpha_m} + \dots + p_1^{\alpha_1-1} p_2^{\alpha_2} p_3^{\alpha_3} \dots p_m^{\alpha_{m-1}}$$
$$+ p_1^{\alpha_1} p_2^{\alpha_2-1} p_3^{\alpha_3-1} p_4^{\alpha_4} \dots p_m^{\alpha_m} + p_1^{\alpha_1} p_2^{\alpha_2-1} p_3^{\alpha_3} p_4^{\alpha_4-1} p_5^{\alpha_5} \dots p_m^{\alpha_m} + \dots + p_1^{\alpha_1} p_2^{\alpha_2-1} p_3^{\alpha_3} \dots p_m^{\alpha_m-1}$$
$$+ \dots + p_1^{\alpha_1} p_2^{\alpha_2} \dots p_{m-1}^{\alpha_{m-1}-1} p_m^{\alpha_m-1})$$

$$+$$

$$\vdots$$

$$+(-1)^{m+1} \left(p_1^{\alpha_1-1} p_2^{\alpha_2-1} \dots p_m^{\alpha_m-1}\right)$$

And the definition of a set G

$$Let\ G = \{g \in (1, \ldots, n) :\ gcd(n, g) \neq 1\}$$

Throughout the course of the proof, it will be shown that expression (A) is equivalent to $|G|$

$$Let\ P_i = \{x \in (1, \ldots, n) :\ p_i\ \mid x\}\ where\ p_i\ is\ a\ prime\ factor\ of\ n$$

Claim: $G = \bigcup_{i=1}^{m} P_i$

Proof:

$$let\ g\ be\ an\ element\ of\ G,\ g \in G$$
$$let\ d = gcd(g, n) \neq 1$$
$$then\ d \mid g\ and\ d \mid n$$
$$as\ d \mid n\ and\ d \neq 1\ \exists p_d \in (p_1, p_2, \ldots, p_m)\ s.t.\ p_d \mid d$$
$$p_d \mid d\ and\ d \mid g\ shows\ p_d \mid g$$
$$then\ g \in P_d$$
$$and\ g \in \bigcup_{i=1}^{m} P_i$$
$$\therefore G \subset \bigcup_{i=1}^{m} P_i$$

$$let\ x\ be\ an\ element\ of\ \bigcup_{i=1}^{m} P_i\ ,\ x \in \bigcup_{i=1}^{m} P_i$$
$$then\ \exists P_x\ s.t.\ x \in P_x$$
$$as\ x \in P_x\ ,\ p_x \mid x$$
$$p_x\ is\ a\ prime\ factor\ of\ n$$
$$so\ p_x \mid n$$
$$as\ p_x \mid x\ and\ p_x \mid n\ then\ gcd(x,n) \geq p_x \neq 1$$
$$thus\ x \in G$$
$$\therefore \bigcup_{i=1}^{m} P_i \subset G$$

$$With\ G \subset \bigcup_{i=1}^{m} P_i\ \ and\ also\ \bigcup_{i=1}^{m} P_i \subset G\ then$$
$$G = \bigcup_{i=1}^{m} P_i$$

Claim: The number of elements in the intersection of any number of P sets is equivalent to the number n divided by each corresponding p.

$\mid P_{i_1} \cap P_{i_2} \cap \ldots \cap P_{i_k} \mid\ =\ n/p_{i_1}p_{i_2}\ldots p_{i_k}$

Proof:

$$Suppose\ x \in\ P_{i_1} \cap P_{i_2} \cap \ldots \cap P_{i_k}$$
$$then,\ p_{i_1} \mid x,\ p_{i_2} \mid x,\ \ldots,\ p_{i_k} \mid x$$
$$as\ p_{i_1}, \ldots, p_{i_k}\ are\ all\ primes\ and\ thus\ relatively\ prime\ to\ each\ other,$$
$$p_{i_1}p_{i_2}\ldots p_{i_k} \mid x$$
$$x = p_{i_1}p_{i_2}\ldots p_{i_k}q$$
$$but\ x \leq n$$
$$qp_{i_1}p_{i_2}\ldots p_{i_k} \leq n$$
$$q \leq n/p_{i_1}p_{i_2}\ldots p_{i_k}$$
$$This\ shows\ n/p_{i_1}p_{i_2}\ldots p_{i_k}\ possible\ solutions\ for\ q$$
$$and\ therefore\ n/p_{i_1}p_{i_2}\ldots p_{i_k}\ possible\ values\ for\ x$$
$$\therefore \mid P_{i_1} \cap P_{i_2} \cap \ldots \cap P_{i_k} \mid\ =\ n/p_{i_1}p_{i_2}\ldots p_{i_k}$$

We can now re-write expression (A) in terms of the size of sets of P

expression (B):

$$\begin{gathered}(\ |P_1|\ +\ |P_2|\ +\ \dots\ +|P_m|)\\ -(|P_1\cap P_2|\ +\ |P_1\cap P_3|\ +\ \dots\ +|P_1\cap P_m|\ +|P_2\cap P_3|+\dots+|P_{m-1}\cap P_m|)\\ +\\ \vdots\\ +(-1)^{m+1}\left(|\bigcap_{i=1}^{m}P_i\ |\right)\end{gathered}$$

Here, we notice the grouping of sets of P in terms of how many possible intersections there are.

Define sets of sets to describe this grouping

$$L_k=\{P_{i_1}\cap P_{i_2}\cap\dots\cap P_{i_k}\}\ for\ k\ =\ 1,\dots m\ and\ P_{i_k}\in(P_1,P_2,\dots,P_m)$$

In other words, that L_k is the set of all possible combinations of exactly k distinct sets of P_i intersected with each other.

Then, we can further simplify (B) to become

expression (C)

$$\sum_{k=1}^{m}(-1)^{k+1}\sum_{i=1}^{|L_k|}\ |l_{k,i}|$$

$$where\ l_{k,i}\ is\ the\ i-th\ distinct\ element\ in\ L_k$$

we now decompose every set $l_{k,i}$ into | $l_{k,i}$ | number of disjoint single element subsets.

$$l_{k,i}\ =\ P_{i_1}\cap P_{i_2}\cap\dots\cap P_{i_k}=\bigcup_{i=1}^{|P_{i_1}\cap P_{i_2}\cap\dots\cap P_{i_k}|}\ S_{p_{i_1}p_{i_2}\dots p_{i_k},i}$$

$$where\ S_{p_{i_1}p_{i_2}\dots p_{i_k},i}=\{s_{p_{i_1}p_{i_2}\dots p_{i_k},i}\}\ ,\ where\ s_{p_{i_1}p_{i_2}\dots p_{i_k},\ i}\ is\ the\ i-th\ element\ of\ P_{i_1}\cap P_{i_2}\cap\dots\cap P_{i_k}$$

As the single element sets are disjoint, then

$$|\ l_{k,i}\ |\ =\ |\ P_{i_1}\cap P_{i_2}\cap\dots\cap P_{i_k}\ |\ =|\ \bigcup_{i=1}^{|P_{i_1}\cap P_{i_2}\cap\dots\cap P_{i_k}|}\ S_{p_{i_1}p_{i_2}\dots p_{i_k},i}\ |\ =\ \sum_{i=1}^{|P_{i_1}\cap P_{i_2}\cap\dots\cap P_{i_k}|}\ |S_{p_{i_1}p_{i_2}\dots p_{i_k},i}\ |$$

Substituting this in on expression (C) we get

expression (D)

$$\sum_{k=1}^{m}(-1)^{k+1}\sum_{i=1}^{|L_k|}\ \sum_{j=1}^{|P_{i_1}\cap P_{i_2}\cap\dots\cap P_{i_k}|}\ |S_{p_{i_1}p_{i_2}\dots p_{i_k},j}\ |$$

when (D) is expanded out, we get the following

$$(|S_{p_1,1}| \ + \ |S_{p_1,2}| + \ldots + |S_{p_1,|P_1|}| \ + \ |S_{p_2,1}| + \ldots + |S_{p_2,|P_2|}| + \ldots + |S_{p_m,|P_m|}|)$$
$$-(|S_{p_1p_2,1}| + |S_{p_1p_2,2}| + \ldots + |S_{p_1p_2,|P_1 \cap P_2|}| + |S_{p_1p_3,1}| + \ldots + |S_{p_1p_3,|P_1 \cap P_3|}| + \ldots + |S_{p_{m-1},\ p_m,\ |P_{m-1} \cap P_m|}|)$$
$$+$$
$$\vdots$$
$$+(-1)^{m+1}(|S_{p_1 \ldots p_m,1}| + \ldots + |S_{p_1 \ldots p_m | \bigcap_{i=1}^{m} P_i|}|$$

However, every set S is a single element subset of G.Then, for an s in a particular S

we have that $s = g$ for some $g \in G$

Define single - element disjoint subsets of G where

$$G_i = \{g_i\} \ for \ i = 1, \ldots, |G| \ where \ g_i \ is \ the \ i-th \ element \ of \ G$$

we can group equivalent sets and then re-write the expanded form of (D) to become

$$(|G_1| + |G_1| + \ldots + |G_1| + |G_2| + \ldots + |G_2| + \ldots + |G_{|G|}| + \ldots + |G_{|G|}|)$$
$$-(|G_1| + |G_1| + \ldots + |G_1| + |G_2| + \ldots + |G_2| + \ldots |G_{|G|}| + \ldots + |G_{|G|}|)$$
$$+$$
$$\vdots$$
$$+(-1)^{m+1}(|G_1| + |G_1| + \ldots + |G_1| + |G_2| + \ldots + |G_2| + \ldots |G_{|G|}| + \ldots + |G_{|G|}|)$$

which is

$$(c_{1,1}|G_1| + c_{1,2}|G_2| + \ldots + c_{1,|G|}|G_{|G|}|)$$
$$-(c_{2,1}|G_1| + c_{2,2}|G_2| + \ldots + c_{2,|G|}|G_{|G|}|)$$
$$+$$
$$\vdots$$
$$+(-1)^{m+1}(c_{m,1}|G_1| + c_{m,2}|G_2| + \ldots + c_{m,|G|}|G_{|G|}|)$$

and simplified to

expression (E)

$$\sum_{i=1}^{|G|} \sum_{k=1}^{m} (-1)^{k+1} c_{k,i} |G_i|$$

where $c_{k,i}$ is the number of times that $|G_i|$ appears in the single element disjoint subset decomposition of L_k

Define a pair of functions,

$$f_1(S,g) = \begin{cases} 1 & if\ g \in S \\ 0 & if\ g \notin S \end{cases}\ where\ S\ is\ a\ set\ of\ numbers,\ and\ g\ \in G$$

$$f_2(L,g) = f_1(l_1,g) + \ldots + f_1(l_{|L|},g)\ where\ L\ is\ a\ set\ of\ sets\ and\ g \in G$$
$$and\ l_i\ is\ the\ i-th\ element\ of\ L$$

then, we can say that $c_{k,i} \ = \ f_2(L_k \ , g_i)$
expression (F)

$$\sum_{i=1}^{|G|}\sum_{k=1}^{m}(-1)^{k+1}f_2(L_k,g_i)$$

For any particular g_i we can say that $gcd(n,g_i) \ = p_{b_1}^{\beta_1}\ldots p_{b_r}^{\beta_r}$ is the prime factorization.

Let L_{k,g_i} be a subset of L_k which is all the sets in L_k that contains g_i

$$L_{k,g_i} = \{S \in L_k :\ g_i \in S\}$$

then, $L_k \backslash L_{k,g_i}$ is the set of sets in L_k that do no contain g_i. So,

$$L_{k,g_i} \ \cup L_k \backslash L_{k,g_i} \ = L_k$$

Take f_2 of both sides with respect to g_i

$$f_2(L_{k,g_i} \cup L_k \backslash L_{k,g_i},\ g_i) = f_2(L_k,g_i)$$
$$and\ as\ L_{k,g_i} \cup L_k \backslash L_{k,g_i} = \emptyset\ then$$
$$f_2(L_{k,g_i},g_i) + f_2(L_k \backslash L_{k,g_i},\ g_i) = f_2(L_k,g_i)$$
$$but,\ \forall S \in L_k \backslash L_{k,g_i}\ ,\ g \notin S,\ then$$
$$f_2(L_k \backslash L_{k,g_i},g_i) = 0,\ and$$
$$f_2(L_{k,g_i},g_i) + 0 = f_2(L_k,g_i)$$
$$f_2(L_{k,g_i},g_i) = f_2(L_k,g_i)$$

If given a set of the intersection of exactly k number of P sets, $P_{i_1} \cap P_{i_2} \cap \ldots \cap P_{i_k}$ then

$$g_i \in P_{i_1} \cap P_{i_2} \cap \ldots \cap P_{i_k}\ only\ if\ \{p_{i_1},\ldots,p_{i_k}\} \subset \{p_{b_1},\ldots p_{b_r}\}$$

That is, from b_r number of sets P_{b_i}, a set $P_{i_1} \cap P_{i_2} \cap \ldots \cap P_{i_k}$ can be constructed by selecting

exactly k number of sets from $\{P_{b_1},\ldots,P_{b_r}\}$ with $g \in P_{i_1} \cap P_{i_2} \cap \ldots \cap P_{i_k}$

The number of potential sets $P_{i_1} \cap P_{i_2} \cap \ldots \cap P_{i_k}$ is equivalent to the combination function of

$$C(b_r,k)$$

then

$$f_2(L_k, g_i) = C(b_r, k)$$

Continuing from expression (F)

$$\sum_{i=1}^{|G|}\sum_{k=1}^{m}(-1)^{k+1}f_2(L_k, g_i)$$

$$\sum_{i=1}^{|G|}\left(\sum_{k=b_r+1}^{m}(-1)^{k+1}f_2(L_k, g_i) + \sum_{k=1}^{b_r}(-1)^{k+1}f_2(L_k, g_i)\right)$$

$$\sum_{i=1}^{|G|}\left(\sum_{k=b_r+1}^{m}(-1)^{k+1}C(b_r, k) + \sum_{k=1}^{b_r}(-1)^{k+1}C(b_r, k)\right)$$

$$but\ when\ k > b_r,\ C(b_r, k) = 0$$

$$\sum_{i=1}^{|G|}\left(\sum_{k=b_r+1}^{m}(-1)^{k+1}(0) + \sum_{k=1}^{b_r}(-1)^{k+1}C(b_r, k)\right)$$

$$\sum_{i=1}^{|G|}\left(0 + \sum_{k=1}^{b_r}(-1)^{k+1}C(b_r, k)\right)$$

expression (G)

$$\sum_{i=1}^{|G|}\sum_{k=1}^{b_r}(-1)^{k+1}C(b_r, k)$$

Claim: $C(a+1, b) = C(a, b) + C(a, b-1)$
Proof:

$$
\begin{aligned}
C(a,b) \; + \; C(a,b-1) &= \frac{a!}{b!(a-b)!} + \frac{a!}{(b-1)!(a-b+1)!} \\
&= \frac{a!}{b(b-1)!(a-b)!} \; + \; \frac{a!}{(a+1+b)(b-1)!(a-b)!} \\
&= \frac{a!}{(b-1)!(a-b)!}\left(\frac{1}{b} + \frac{1}{a+1-b}\right) \\
&= \frac{a!}{(b-1)!(a-b)!}\left(\frac{a+1-b}{b(a+1-b)} + \frac{b}{b(a+1-b)}\right) \\
&= \frac{a!}{(b-1)!(a-b)!}\left(\frac{a+1}{b(a+1-b)}\right) \\
&= \frac{(a+1)a!}{b(b-1)!(a+1-b)(a-b)!} \\
&= \frac{(a+1)!}{b!(a+1-b)!} \\
&= C(a+1,b)
\end{aligned}
$$

Claim: $\sum_{i=1}^{a}(-1)^{i+1}C(a,i) = 1$ for any a

Proof:

$$
\begin{aligned}
\sum_{i=1}^{a}(-1)^{i+1}C(a,i) \; &= C(a,1) - C(a,2) + C(a,3) + \ldots + (-1)^{a}C(a,a-1) + (-1)^{a+1}C(a,a) \\
&= (C(a-1,0) + C(a-1,1)) \; - \; (C(a-1,1) + C(a-1,2)) \; + \; (C(a-1,2) + C(a-1,3)) - \ldots \\
&\quad + (-1)^{a}(C(a-1,a-2) + C(a-1,a-1)) + (-1)^{a+1}(C(a,a)) \\
&= C(a-1,0) + (C(a-1,1) - C(a-1,1)) \; + \; (C(a-1,2) - C(a-1,2)) + \ldots \\
&\quad + (-1)^{a}C(a-1,a-1) + (-1)^{a+1}C(a,a) \\
&= C(a-1,0) + (-1)^{a}C(a-1,a-1) + (-1)^{a+1}C(a,a) \\
&= 1 \; + \; (-1)^{a} \; (1) + (-1)(-1)^{a}(1) \\
&= 1 \; + \; (-1)^{a} - (-1)^{a} \\
&= 1 \; + \; 0 \\
&= 1
\end{aligned}
$$

From expression (G)

$$\sum_{i=1}^{|G|}\sum_{k=1}^{b_r}(-1)^{k+1}C(b_r,k)$$

$$but\ \sum_{k=1}^{b_r}(-1)^{k+1}C(b_r,k)\ =\ 1$$

$$\sum_{i=1}^{|G|}1$$

expression (H)

$$|G|$$

we have that $(A)\ = (H)$.

The totient function, $\varphi(n)$ is the number of integers less than or equal to n that are relatively prime to n

As G is the set of numbers that are less than or equal to n that are not relatively prime to n, then

$$\varphi(n)+|G|=n$$
$$\varphi(n)=n-|G|$$

And finally

$$\varphi(n)=n$$

$$-\ \left(p_1^{\alpha_1-1}p_2^{\alpha_2}\dots p_m^{\alpha_m}\ +p_1^{\alpha_1}p_2^{\alpha_2-1}p_3^{\alpha_3}\dots p_m^{\alpha_m}+\dots+p_1^{\alpha_1}p_2^{\alpha_2}\dots p_m^{\alpha_m-1}\right)$$

$$+(p_1^{\alpha_1-1}p_2^{\alpha_2-1}p_3^{\alpha_3}\dots p_m^{\alpha_m}+p_1^{\alpha_1-1}p_2^{\alpha_2}p_3^{\alpha_3-1}p_4^{\alpha_4}\dots p_m^{\alpha_m}+\dots+p_1^{\alpha_1-1}p_2^{\alpha_2}p_3^{\alpha_3}\dots p_m^{\alpha_{m-1}}$$
$$+p_1^{\alpha_1}p_2^{\alpha_2-1}p_3^{\alpha_3-1}p_4^{\alpha_4}\dots p_m^{\alpha_m}+p_1^{\alpha_1}p_2^{\alpha_2-1}p_3^{\alpha_3}p_4^{\alpha_4-1}p_5^{\alpha_5}\dots p_m^{\alpha_m}+\dots+p_1^{\alpha_1}p_2^{\alpha_2-1}p_3^{\alpha_3}\dots p_m^{\alpha_m-1}$$
$$+\dots+p_1^{\alpha_1}p_2^{\alpha_2}\dots p_{m-1}^{\alpha_{m-1}-1}p_m^{\alpha_m-1})$$

$$-$$

$$\vdots$$

$$+(-1)^m\left(p_1^{\alpha_1-1}p_2^{\alpha_2-1}\dots p_m^{\alpha_m-1}\right)$$

□

Summary of Proof

$$\left(p_1^{\alpha_1-1}p_2^{\alpha_2}\dots p_m^{\alpha_m} + p_1^{\alpha_1}p_2^{\alpha_2-1}p_3^{\alpha_3}\dots p_m^{\alpha_m} + \dots + p_1^{\alpha_1}p_2^{\alpha_2}\dots p_m^{\alpha_m-1}\right)$$

$$-(p_1^{\alpha_1-1}p_2^{\alpha_2-1}p_3^{\alpha_3}\dots p_m^{\alpha_m} + p_1^{\alpha_1-1}p_2^{\alpha_2}p_3^{\alpha_3-1}p_4^{\alpha_4}\dots p_m^{\alpha_m} + \dots + p_1^{\alpha_1-1}p_2^{\alpha_2}p_3^{\alpha_3}\dots p_m^{\alpha_{m-1}}$$
$$+p_1^{\alpha_1}p_2^{\alpha_2-1}p_3^{\alpha_3-1}p_4^{\alpha_4}\dots p_m^{\alpha_m} + p_1^{\alpha_1}p_2^{\alpha_2-1}p_3^{\alpha_3}p_4^{\alpha_4-1}p_5^{\alpha_5}\dots p_m^{\alpha_m} + \dots + p_1^{\alpha_1}p_2^{\alpha_2-1}p_3^{\alpha_3}\dots p_m^{\alpha_m-1}$$
$$+ \dots + p_1^{\alpha_1}p_2^{\alpha_2}\dots p_{m-1}^{\alpha_{m-1}-1}p_m^{\alpha_m-1})$$

$$+$$
$$\vdots$$

$$+(-1)^{m+1}\left(p_1^{\alpha_1-1}p_2^{\alpha_2-1}\dots p_m^{\alpha_m-1}\right)$$

$$= (\ |P_1| + |P_2| + \dots + |P_m|)$$
$$-(|P_1 \cap P_2| + |P_1 \cap P_3| + \dots + |P_1 \cap P_m| + |P_2 \cap P_3| + \dots + |P_{m-1} \cap P_m|)$$
$$+$$
$$\vdots$$
$$+(-1)^{m+1}\left(|\bigcap_{i=1}^{m} P_i\ |\right)$$

$$= \sum_{k=1}^{m}(-1)^{k+1}\sum_{i=1}^{|L_k|}\ |l_{k,i}|$$

$$= \sum_{k=1}^{m}(-1)^{k+1}\sum_{i=1}^{|L_k|}\sum_{j=1}^{|P_{i_1}\cap P_{i_2}\cap\dots\cap P_{i_k}|}|S_{p_{i_1}p_{i_2}\dots p_{i_k},j}\ |$$

$$= \sum_{i=1}^{|G|}\sum_{k=1}^{m}(-1)^{k+1}c_{k,i}|G_i|$$

$$= \sum_{i=1}^{|G|}\sum_{k=1}^{m}(-1)^{k+1}f_2(L_k, g_i)$$

$$= \sum_{i=1}^{|G|}\sum_{k=1}^{b_r}(-1)^{k+1}C(b_r, k)$$
$$= |G|$$