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HW#7 Problem 5

Let $k \leq n$. If $(1, \ldots, k)$ is a cycle of length k in S_n and σ is a cycle of length k in S_n, then there is a transposition τ so that $\tau\sigma\tau^{-1} = (1, \ldots, k)$.

The problem is erroneous and has a counterexample

$$suppose\ there\ exists\ a\ transposition\ \tau\ such\ that$$
$$(1, 2, 3) = \tau\sigma\tau^{-1}\ where\ \sigma = (3, 4, 5)$$

$$let\ \tau = (s_1, s_2)$$
$$as\ (1, 2, 3) = \tau\sigma\tau^{-1},\ then\ \tau\sigma\tau^{-1}(2) = 3$$

$$suppose\ 2 \notin (s_1, s_2)$$
$$then,\ \tau(2) = 2,\ \sigma(2) = 2,\ \tau^{-1}(2) = 2$$
$$and\ \tau\sigma\tau^{-1}(2) = 2\ \neq 3$$
$$then\ the\ assumption\ 2 \notin (s_1, s_2)\ is\ false$$
$$then\ 2 \in (s_1, s_2)$$

$$as\ (1, 2, 3) = \tau\sigma\tau^{-1},\ then\ \tau\sigma\tau^{-1}(1) = 2$$
$$suppose\ 1 \notin (s_1, s_2)$$
$$then,\ \tau(1) = 1,\ \sigma(1) = 1,\ \tau^{-1}(1) = 1$$
$$and\ \tau\sigma\tau^{-1}(1) = 1 \neq 2$$
$$then\ the\ assumption\ 1 \notin (s_1, s_2)\ is\ false$$
$$then\ 1 \in (s_1, s_2)$$

$$as\ 1, 2\ \in (s_1, s_2)$$
$$then\ \tau = (1, 2)$$

$$but,\ \tau\sigma\tau^{-1}(1) = \tau\sigma(2) = \tau(2) = 1 \neq 2$$

$$therefore,\ the\ assumption\ \tau\ is\ a\ transposition\ is\ false$$

However, if we modify the problem, then it always has a solution

Let $k \leq n$. If $(1, \ldots, k)$ is a cycle of length k in S_n and σ is a cycle of length k in S_n, then there is a **permutation** τ so that $\tau\sigma\tau^{-1} = (1, \ldots, k)$.

$$let\ \phi = (1, \dots, k)\ and\ \sigma = \big(x, \sigma(x), \sigma^2(x), \dots, \sigma^{k-1}(x)\big)$$
$$split\ \{1, \dots, n\}\ into\ 4\ groups$$
$$A = \{a_1, \dots, a_e\}\ where\ a_i \in \phi,\ a_i \notin \sigma,\ for\ i = 1, \dots, e$$
$$B = \{b_1, \dots, b_f\}\ where\ b_i \in \phi,\ b_i \in \sigma,\ for\ i = 1, \dots, f$$
$$C = \{c_1, \dots, c_g\}\ where\ c_i \notin \phi,\ c_i \in \sigma,\ for\ i = 1, \dots, g$$
$$D = \{d_1, \dots, d_h\}\ where\ d_i \notin \phi,\ d_i \notin \sigma,\ for\ d = 1, \dots, h$$

$$Note\ that\ e = g\ as$$
$$(g + h) + k = n = (e + h) + k$$
$$in\ other\ words,\ the\ number\ of\ elements\ not\ in\ \phi,\ which\ is\ (g + h)$$
$$plus\ the\ number\ of\ elements\ in\ \phi,\ which\ is\ k,\ is\ equal\ to\ n.$$
$$Also,\ the\ number\ of\ elements\ not\ in\ \sigma,\ which\ is\ (e + h)$$
$$plus\ the\ number\ of\ elements\ in\ \sigma,\ which\ is\ k,\ is\ also\ equal\ to\ n.$$

$$construct\ \tau^{-1}\ by\ the\ following$$

$$if\ s \in D,\ then\ define\ \tau^{-1}(s) = s$$

$$if\ s \in C,\ then\ \exists\ i\ \le g\ s.t.\ s = c_i$$
$$then,\ define\ \tau^{-1}(s) = a_i$$

$$if\ s \in B\ or\ s \in A,\ then\ \exists m \le k\ s.t.\ \phi^m(1) = s$$
$$then,\ define\ \tau^{-1}(s) = \sigma^m(x)$$

$$from\ these\ rules,\ we\ get$$
$$\tau^{-1} = \begin{pmatrix} 1 & 2 & \dots & k & c_1 & \dots & c_g & d_1 & \dots & d_n \\ x & \sigma(x) & \dots & \sigma^{k-1}(x) & a_1 & \dots & a_e & d_1 & \dots & d_n \end{pmatrix}$$

$$\tau = \begin{pmatrix} x & \sigma(x) & \dots & \sigma^{k-1}(x) & a_1 & \dots & a_e & d_1 & \dots & d_n \\ 1 & 2 & \dots & k & c_1 & \dots & c_g & d_1 & \dots & d_n \end{pmatrix}$$

$$finally,\ \forall s \in \{1,\ldots,n\}\ three\ cases\ exist$$
$$either\ s \in D,\ s \in C,\ or\ s \in A \cup B$$

$$in\ the\ case\ of\ s \in D$$
$$\tau\sigma\tau^{-1}(s) = \tau\sigma(s) = \tau(s) = s = \phi(s)$$

$$in\ the\ case\ of\ s \in C$$
$$for\ some\ particular\ a \in A,\ then$$
$$\tau\sigma\tau^{-1}(s) = \tau\sigma(a) = \tau(a) = s = \phi(s)$$

$$in\ the\ case\ of\ s \in A \cup B$$
$$then,\ s = \phi^m(1)\ for\ some\ m \leq k\ and$$
$$if\ m < k$$
$$\tau\sigma\tau^{-1}(s) = \tau\sigma\tau^{-1}\left(\phi^m(1)\right) = \tau\sigma\left(\sigma^m(x)\right) = \tau\left(\sigma^{m+1}(x)\right) = m+1 = \phi^{m+1}(1) = \phi\left(\phi^m(1)\right) = \phi(s)$$
$$if\ m = k$$
$$\tau\sigma\tau^{-1}(s) = \tau\sigma\tau^{-1}\left(\phi^k(1)\right) = \tau\sigma\left(\sigma^k(x)\right) = \tau(\sigma(x)) = 2 = \phi(1) = \phi\left(\phi^k(1)\right) = \phi(s)$$

$$thus,\ in\ all\ cases,\ \tau\sigma\tau^{-1}(s) = \phi(s) = (1,\ldots,k)(s)$$