February 7, 2019 Homework 1 due February 14, 2019
Solutions

1. Prove that the following holds for each two sets A and B

$$\overline{A \cup B} = \overline{A} \cap \overline{B}.$$

Solution.

a) $x \in \overline{A \cup B} \Rightarrow x \notin A \cup B \Rightarrow (x \notin A \text{ and } x \notin B) \Rightarrow (x \in \overline{A} \text{ and } x \in \overline{B}) \Rightarrow x \in \overline{A} \cap \overline{B}$. This shows $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$.

b) $x \in \overline{A} \cap \overline{B} \Rightarrow (x \in \overline{A} \text{ and } x \in \overline{B}) \Rightarrow (x \notin A \text{ and } x \notin B) \Rightarrow x \notin A \cup B \Rightarrow x \in \overline{A \cup B}$. This shows $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$ and completes the proof.

2. Build a bijection $f : \mathbf{N} \to \mathbf{N}'$, where $\mathbf{N}'$ is a proper subset of $\mathbf{N}$.

Solution. Let $f(n) = S(n)$.

3. Show that $(x + y) + z = x + (y + z)$.

Solution. Note that $(x + y) + 0 = x + y = x + (y + 0)$. Assume now that

$$(x + y) + z = x + (y + z)$$

holds for some $z \in \mathbf{N}$. Note that

$$(x + y) + S(z) = S((x + y) + z) = S((x + (y + z)) = x + S(y + z) = x + (y + S(z)).$$

4. Show that if $x + z = y + z$, then $x = y$.

Solution. With $z = 0$ one has $x = x + 0 = y + 0 = y$. Assume that $x + z = y + z$ implies $x = y$. Note that $x + S(z) = S(x + z)$ and $y + S(z) = S(y + z)$. Hence if $x + S(z) = y + S(z)$, then $S(x + z) = S(y + z)$, and $x + z = y + z$. This yields $x = y$.

Here are results that will be used for the next problems:

a) $S(n)m = nm + m$ (see class material)

b) $nS(m) = nm + n$

proof. First note that one has $0 \cdot S(m) = 0$ (see class material). Assume that $n \cdot S(m) = nm + n$ for some $n \in \mathbf{N}$. We next will show that $S(n) \cdot S(m) = S(n) \cdot m + S(n) = nm + m + S(n)$.

Look at $\psi(n) = n + S(m)$, and f so that $f(0) = 0$, and $\psi(f(k)) = f(S(k))$ for each $k \in \mathbf{N}$. Note that

$$S(n) \cdot S(m) = f(S(n)) = \psi(f(n)) = f(n) + S(m) = n \cdot S(m) + S(m) = nm + n + S(m).$$

In addition one has (see class material)

$$nm + [m + S(n)] = nm + [n + S(m)].$$

This completes the proof.

5. Show that $(xy)z = x(yz)$.

Solution. Note that $(xy)0 = 0 = x(y0)$. Assume $(xy)z = x(yz)$ holds for some z. Then

$$(xy)S(z) = (xy)z + xy = x(yS(z) + y) = x(yS(z)).$$

To complete the proof invoke Peano axiom 3.

6. Show that $xy = yx$.

Solution. Note that $0y = 0 = y0$ (see class material). If $xy = yx$, then $S(x)y = xy + y = yS(x)$. To complete the proof invoke Peano axiom 3.

7. Show that if $xz = yz$ and $z \neq 0$, then $x = y$.

Solution. Assume the contrary, i.e. $x \neq y$. In this case $x > y$, or $y > x$. If $x > y$, then $xz > yz$. This contradiction shows that $x > y$ is false. The case $y > x$ can be handled similarly. The assumption $x \neq y$ leads to a contradiction, hence it is false, and $x = y$.

8. Show that $x(y + z) = xy + xz$.

Solution. Note that $0(y + z) = 0y + 0z$. If $x(y + z) = xy + xz$ for some x, then consider $S(x)(y + z)$ and $S(x)y + S(x)z$.

$$S(x)(y + z) = x(y + z) + (y + z) = (xy + y) + (xz + z) = S(x)y + S(x)z.$$