February 14, 2019 Homework 2 due February 21, 2019
Solutions

1. Show that for each $n \in \mathbf{N}$ which is different from 0 there is $m \in \mathbf{N}$ such that $S(m) = n$.

   Solution. Let $\mathbf{N}'$ be a subset of $\mathbf{N}$ defined as follows:

$$\mathbf{N}' = \{0, \text{ and } n \in \mathbf{N} \text{ so that there is } m \in \mathbf{N} \text{ with } S(m) = n\}.$$

   We note that $0 \in \mathbf{N}'$, and if $n \in \mathbf{N}'$, then $S(n) \in \mathbf{N}'$. Hence $\mathbf{N}' = \mathbf{N}$. This completes the proof.

2. Let a, b, c, and $d \in \mathbf{N}$. Show that if $a \geq b$ and $c \geq d$, then $a + c \geq b + d$.

   Solution. Since $a \geq b$ one has $a + c \geq b + c = c + b$. Since $c \geq d$ one has $c + b \geq d + b$.

3. Let $\mathbf{N} = \{0, 1\}$ and define $S(0) = 1$, and $S(1) = 1$. Show that $\mathbf{N}$ satisfies Peano's axioms 1 and 3, but not 2. Let $\psi : \mathbf{N} \to \mathbf{N}$ such that $\psi(0) = 1$, and $\psi(1) = 0$. Show that the recursion theorem breaks down for this ψ. That is there exists no map $f : \mathbf{N} \to \mathbf{N}$ such that $f(0) = 0$, and $f(S(n)) = \psi(f(n))$.

   Solution. If f exists, then

$$f(1) = f(S(0)) = \psi(f(0)) = \psi(0) = 1,$$
and
$$f(1) = f(S(1)) = \psi(f(1)) = \psi(1) = 0.$$

   This contradiction completes the proof.

4. If x, y, and $z \in \mathbf{Z}$ show that

   a) $(x + y)z = xz + yz$,

      Solution. Straightforward computation.

   b) if $z \neq 0$, then $xz = yz$ implies $x = y$.

      Solution. Select $(n, 0)$, $n \in \mathbf{N}$ as a representative of the integer $z \geq 0$. If (a_x, b_x) is a representative for x, and (a_y, b_y) is a representative for y, then $xz = yz$ yields $na_x + nb_y \geq nb_x + na_y$ and $a_x + b_y \geq b_x + a_y$. This shows $x \geq y$, and completes the proof.

c) if $x \geq y$ and $z \geq 0$, then $xz \geq yz$.

Solution. Select $(n, 0)$, $n \in \mathbf{N}$ as a representative of the integer $z \geq 0$. If (a_x, b_x) is a representative for x, and (a_y, b_y) is a representative for y, then $x \geq y$ yields $a_x + b_y \geq b_x + a_y$. Note that $zx = (na_x, nb_x)$ and $zy = (na_y, nb_y)$. Since $n(a_x + b_y) \geq n(b_x + a_y)$ one has $zx \geq zy$.

d) if $x \geq y$, then $x + z \geq y + z$.

Solution. Straightforward computation.

e) If $x \neq y$, then $x > y$ or $y > x$.

Solution. Follows straightforward from definition.