March 7, 2019 Homework 4 due March 14, 2019
Solutions

1. True or False? n^2+1 is not divisible by 11 for each $n \in Z$.

   Solution. Let $n = 11q + n_1$ with $0 \le n_1 < 11$. Since $n^2 \equiv n_1^2 \pmod{11}$ we have to check the statement for $n_1 = 0, 1, \ldots, 10$ only.

n_1	n_1^2+1
0	1
1	2
2	5
3	10
4	17
5	26
6	37
7	50
8	65
9	82
10	101

2. $21(n^2+1)$ is not divisible by 11 for each $n \in Z$.

   Solution. Since 11 is a prime number if $p|21(n^2+1)$, then $p|(n^2+1)$. The result now follows from the previous Problem.

3. Let $n = 8k + 7$. True or False? There are integers a, b, and c so that $n = a^2 + b^2 + c^2$.

   Solution. Assume the integers a, b, and c exist. Then $a = 8q_a + r_a$, $b = 8q_b + r_b$, and $c = 8q_c + r_c$, with $0 \le r_a, r_b, r_c < 8$. Moreover $a^2+b^2+c^2 \equiv r_a^2 + r_b^2 + r_c^2 \pmod 8$. Since the only possible values for r_a, r_b, r_c are $0, 1, \ldots, 7$ the only possible values for $r_a^2, r_b^2, r_c^2 \pmod 8$ are $0, 1$ and 4. A straightforward analysis of these cases shows that 7 may not be the remiainder for $a^2 + b^2 + c^2$.

4. Solve $x^4 + x^3 + x^2 + x + 1 \equiv 0 \pmod 2$.

   Solution. Note that $x^4 + x^3 + x^2 + x + 1$ is odd when x is even, and $x^4 + x^3 + x^2 + x + 1$ is also odd when x is odd. Hence no integer x solves the congruence.

5. Problem True or False? If p is a prime, and $a, b > 1$, then $(a+b)^p \equiv a^p + b^p \pmod p$.

   Solution. Note that $(a+b)^p = a^p + \sum_{n=1}^{p-1} \binom{p}{n} a^{p-n}b^n + b^p$. For $1 < n < p$

$$\binom{p}{n} = \frac{p!}{n!(p-n)!} = \frac{(n+1)(n+2)\cdots(p-1)p}{1\cdot 2\cdots(p-n)} = \frac{(n+1)(n+2)\cdots(p-1)}{1\cdot 2\cdots(p-n)}p.$$

Since p is prime $(n+1)(n+2)\cdots(p-1) = d(p-n)!$, and $\begin{pmatrix} p \\ n \end{pmatrix} a^{p-n}b^n \equiv 0 \pmod{p}$.

6. Find all integers n such that $3n+7$ is divisible by 11.

   Solution. $n = 11q + n_1$ with $0 \le n_1 < 11$. A straightforward computation shows that $3n_1 + 7$ is divisible by 11 only when $n_1 = 5$. Hence $n \equiv 5 \pmod{11}$ are the solutions.

7. Show that $10^{2n} = 11q + 1$, and $10^{2n+1} = 11q - 1$.

   Solution. Use induction.

8. Consider a k digit integer $n = n_{k-1} \ldots n_1 n_0$. True or False? If $\sum_{i \text{ is even}} n_i \equiv \sum_{i \text{ is odd}} n_i \pmod{11}$, then $11|n$.

   Solution. $n = n_{k-1}n_k \ldots n_1 n_0 = n_0 + n_1 10^1 + n_2 10^2 + \cdots + n_{k-1}10^{(k-1)}$. Use the previous Problem.