April 11, 2019 Homework 7 due April 25, 2019
Solutions

1. True or False? If $\sigma, \tau \in S_n$, then $\sigma\tau = \tau\sigma$.

   Solution. Let
$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \text{ and } \tau = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$
   Note that
$$\tau\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \text{ and } \sigma\tau = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

2. If s_1, s_2, s_3, and s_4 are distinct, then
$$(s_2, s_3)(s_1, s_4) = (s_1, s_4)(s_2, s_3), \text{ and } (s_2, s_3)(s_1, s_3) = (s_1, s_2)(s_2, s_3).$$

   Solution. Since s_1, s_2, s_3, and s_4 are distinct the cycles (s_2, s_3) and (s_1, s_4) are disjoint. This yields the first result. Let σ be defined as
$$\sigma(s) = \begin{cases} s_2 & \text{if } s = s_1 \\ s_3 & \text{if } s = s_2 \\ s_1 & \text{if } s = s_3 \end{cases}$$
   Note that $(s_2, s_3)(s_1, s_3) = \sigma = (s_1, s_2)(s_2, s_3)$.

3. Let $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$. Represent σ as a product of transpositions.

   Solution. First note that any $\tau \in S_2$ is a transposition. It suffices then to find a transposition δ so that $\delta(\sigma(3)) = 3$. If $\delta = (1, 3)$, then $\delta \cdot \sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$, and $(1, 2)(1, 3)\sigma = (1)$, or $\sigma = (1, 3)^{-1}(1, 2)^{-1} = (1, 3)(1, 2)$.

4. Note that $(1, 2)$ is an odd permutation. Indeed $(1, 2)(1, 2) = (1)$. If σ is a cycle of length 3, and $\sigma \in S_3$, then $\tau = (3, \sigma(3))\sigma$ is a cycle of length 2, hence τ is an even permutation. This shows that σ is an odd permutation. Show that

   a) if $\sigma \in S_{2n}$ is a cycle of length $2n$, then σ is an even permutation.

   b) if $\sigma \in S_{2n+1}$ is a cycle of length $2n + 1$, then σ is an odd permutation.

   Solution. The claim is false as stated. However if "even" and "odd" are interchanged in a) and b) above it becomes true. Indeed, note that the statement holds true for $\sigma \in S_2$. Assume that the statement holds true for any $\sigma \in S_k$. If $\sigma = (s_1, \ldots, s_k, k + 1)$ is a cycle in S_{k+1}, then $\tau = (k + 1, s_1)\sigma = (s_1, \ldots, s_k)$. This shows that the statement holds for the cycle σ, and completes the proof.

5. Let $k \le n$. If $(1, \ldots, k)$ is a cycle of length k in S_n, and σ is a cycle of length k in S_n, then there is a transposition τ so that $\tau\sigma\tau^{-1} = (1, \ldots, k)$.

   Solution. Let

$$\sigma = \begin{pmatrix} s_1 & \ldots & s_k & t_1 & \ldots & t_{n-k} \\ s_2 & \ldots & s_1 & t_1 & \ldots & t_{n-k} \end{pmatrix}, \text{ and } \tau = \begin{pmatrix} s_1 & \ldots & s_k & t_1 & \ldots & t_{n-k} \\ 1 & \ldots & k & k+1 & \ldots & n \end{pmatrix}.$$

   A straightforward computation shows that $\tau\sigma\tau^{-1} = (1, \ldots, k)$.

6. Let $k \le n$, and $\sigma_n \in S_n$ such that $\sigma_n(i) = i$ when $i > k$. We shall say that σ_n is associated with $\sigma_k \in S_k$ in the obvious way if $\sigma_k(i) = \sigma_n(i)$, $i = 1, \ldots, k$. For σ_n, $\tau_n \in S_n$ define $\sigma_n \sim \tau_n$ if $\sigma_n\tau_n^{-1} \in S_k$. For a given $\sigma_n \in S_n$ describe all $\tau_n \in S_n$ so that $\sigma_n\tau_n^{-1} \in S_k$.

   Solution. Let τ_n be a permutation so that τ_n^{-1} maps $\{k+1, \ldots, n\}$ to $\{\sigma_n^{-1}(k+1), \ldots, \sigma_n^{-1}(n)\}$. Note that for $i \in \{k+1, \ldots, n\}$ one has

$$\tau_n^{-1}(i) \in \{\sigma_n^{-1}(k+1), \ldots, \sigma_n^{-1}(n)\}, \text{ and } \sigma_n\tau_n^{-1}(i) \in \{k+1, \ldots, n\}.$$

   Hence $\sigma_n\tau_n^{-1} \in S_k$ in the obvious way.

   On the other hand, if $\tau_n^{-1}(j) = \sigma_n^{-1}(i)$ for $k+1 \le j \le n$, and $1 \le i \le k$, then

$$\sigma_n\left(\tau_n^{-1}(j)\right) = \sigma_n\left(\sigma_n^{-1}(i)\right) = i,$$

   and $\sigma_n\tau_n^{-1} \notin S_k$ in the obvious way.