April 26, 2019 Homework 8 due May 9, 2019
Solutions

1. Let let G be a group. If for each $a \in G$ one has $a^2 = 1$, then for all $a, b \in G$ one has $ab = ba$.

   Solution. Note that $ab(ab) = 1 = ab(ba)$. This shows that $(ab)^{-1} = ab$, and also $(ab)^{-1} = ba$. Since the inverse unique, the result follows.

2. Let $G = \mathbf{R} \setminus \{-1\}$. Define $a * b = a + b + ab$. Show that $(G, *)$ is a group.

   Solution. Note that $a * 0 = a$, and $a * \left(\frac{-a}{1+a}\right) = 0$.

3. Let $y_a a = a$ and $y_b b = b$. Show that $y_a = y_b$.

   Solution. Let $b = ay$, then $y_a b = y_a(ay) = (y_a a)y = ay = b$.

4. Show that the set $\{f_{m,b} : \mathbf{R} \to \mathbf{R} : m \neq 0, \text{ and } f_{m,b}(x) = mx + b\}$ forms a group under composition of functions.

   Solution. Note that $f_{n,a}(f_{m,b}(x)) = f_{nm,nb+a}(x)$. Hence $f_{1,0}(f_{m,b}) = f_{m,b} = f_{m,b}(f_{1,0})$. Also wnen $n = \frac{1}{m}$, and $a = -\frac{b}{m}$ one has $f_{n,a}(f_{m,b}(x)) = f_{m,b}(x) = f_{m,b}(f_{n,a}(x))$.

5. Show that the set of matrices $\begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix}$ with $m \neq 0$ is a group under matrix multiplication.

   Solution. Note that

   $$\begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} n & a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} mn & ma + b \\ 0 & 1 \end{bmatrix}.$$

   When $n = \frac{1}{m}$, and $a = -\frac{b}{m}$ one has

   $$\begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} n & a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} n & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix}.$$

6. Let G be a group, and $|G| = 2n$. True or False? There is $a \in G$ so that $a \neq 1$, and $a^2 = 1$.

   Solution. Suppose the opposite, that is for each $a \neq 1$ the equation $ab = 1$ implies $a \neq b$. The sets $\{a, b\}$, $\{c, d\}$ if not identical, then disjpoint. This shows that $|G|$ is odd. This contradiction completes the proof.

7. Let S be a subset of $GL_2(\mathbf{R})$ that consists of symmetric matrices. True or False? S is a subgroup of $GL_2(\mathbf{R})$.

   Solution. Consider $a, b \in S$

   $$a = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \text{ and } b = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}. \text{ Note that } ab = \begin{bmatrix} 2 & 4 \\ 3 & 7 \end{bmatrix} \notin S.$$

8. Let $a = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}$. Show that $a^n = \begin{bmatrix} f_{n+1} & -f_n \\ -f_n & f_{n-1} \end{bmatrix}$, where $f_0 = 0$, $f_1 = 1$, and $f_{n+1} = f_n + f_{n-1}$ is the Fibonacci sequence.

Solution. Use induction.