Project, due date: Sunday, May 19, 2019

1. Let $n \in \mathbf{N}$. True or False? There exists $m \in \mathbf{N}$ such that $m > n$ and $S(n) > m$.

2. Let $a_i \geq 0$, $i = 1, \ldots, n$, $A_n = \dfrac{1}{n}(a_1 + \cdots + a_n)$, and $G_n = (a_1 a_2 \cdot \cdots \cdot a_n)^{1/n}$. Show that

   a) $G_{2^n} \leq A_{2^n}$, $n = 1, 2, \ldots$.

   b) $G_n \leq A_n$.

3. Let $\gcd(a, b) = 1$. True or False? For $n > 1$ one has $\gcd(a^n, b) = 1$.

4. Find all solutions x, y for the equation $ax + by = c$, where $a, b, c, x, y \in \mathbf{Z}$.

5. If $gcd(n, m) = 1$, then $n^{\varphi(m)} + m^{\varphi(n)} \equiv 1 \mod nm$.

6. a) Let r, n be integers. Show that $1 + r + \cdots + r^n = \begin{cases} n+1 & \text{if} \quad r = 1, \\ \dfrac{r^{n+1} - 1}{r - 1} & if \quad r \neq 1. \end{cases}$

   b) Let p, n be integers. If p is a prime, then $p^n = \varphi(1) + \varphi(p) + \varphi(p^2) + \cdots + \varphi(p^n)$.

   c) Let n be an integer. Show that $\sum_{d|n} \varphi(d) = n$.

7. If $n > 1$, then the sum of all positive integers less that n and relatively prime to n is $\dfrac{n\varphi(n)}{2}$.

8. Note that for $n = 2$ the set S_n equipped with composition becomes a cyclic group, indeed for $\sigma = (1, 2)$ one has $\sigma^2 = (1)$, and $S_n = \{\sigma, \sigma^2\}$. True or False? S_n is a cyclic group when $n > 2$.

9. Let G be a finite group, and $n > 2$. Let A_n be the set of elements of G that have order n. True or False? $|A_n|$ is an even number.