MATH411
quiz 0
02/04/20
Total 100
Solutions

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Show all work legibly. **Name**:______________________

1. (30) Let $\mathbf{u}_1,\ \mathbf{u}_2$ be linearly independent vectors of magnitude 1 (i.e., $\mathbf{u}_1^T\mathbf{u}_1 = \mathbf{u}_2^T\mathbf{u}_2 = 1$).

   (a) (10) True or False? $\left|\mathbf{u}_1^T\mathbf{u}_2\right| \le 1$

   Solution. Note that for each real number t

   $$0 \le (\mathbf{u}_1 - t\mathbf{u}_2)^T(\mathbf{u}_1 - t\mathbf{u}_2) = t^2\mathbf{u}_2^T\mathbf{u}_2 - 2t\mathbf{u}_2^T\mathbf{u}_1 + \mathbf{u}_1^T\mathbf{u}_1 = t^2 - t\left(2\mathbf{u}_2^T\mathbf{u}_1\right) + 1.$$

   This yields

   $$\left(2\mathbf{u}_2^T\mathbf{u}_1\right)^2 - 4 \le 0, \text{ and } \left|\mathbf{u}_1^T\mathbf{u}_2\right| \le 1.$$

   Mark one and explain.
   ▫ True ▫ False

   (b) (10) True or False? If $\left|\mathbf{u}_1^T\mathbf{u}_2\right| = 1$, then $\mathbf{u}_1 = \pm\mathbf{u}_2$

   Solution. When $\mathbf{u}_1^T\mathbf{u}_2 = 1$ one has $|\mathbf{u}_1 - t\mathbf{u}_2|^2 = (t-1)^2$. When $t = 1$ one has $|\mathbf{u}_1 - \mathbf{u}_2|^2 = 0$, and $\mathbf{u}_1 = \mathbf{u}_2$. If $\mathbf{u}_1^T\mathbf{u}_2 = -1$, one has $|\mathbf{u}_1 - t\mathbf{u}_2|^2 = (t+1)^2$. For $t = -1$ this yields $|\mathbf{u}_1 + \mathbf{u}_2|^2 = 0$, and $\mathbf{u}_1 = -\mathbf{u}_2$. Note that the conditions

   i. $\mathbf{u}_1,\ \mathbf{u}_2$ are linearly independent vectors of magnitude 1,
   ii. $\left|\mathbf{u}_1^T\mathbf{u}_2\right| = 1$

   are mutually exclusive.

   Mark one and explain.
   ▫ True ▫ False

(c) (10) Let $\mathbf{v}_1,\ \mathbf{v}_2$ be linearly independent vectors of magnitude 1. True or False? If $\mathbf{u}_i^T\mathbf{v}_j = 0$ for each i, j then the set $\{\mathbf{u}_1,\ \mathbf{u}_2,\ \mathbf{v}_1,\ \mathbf{v}_2\}$ is linearly independent.

Solution. Consider the equation

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + d_1\mathbf{v}_1 + d_2\mathbf{v}_2 = 0.$$

The dot product of $\mathbf{u}_1$ with this equation yields $c_1 + c_2\mathbf{u}_1^T\mathbf{u}_2 = 0$. The dot product of $\mathbf{u}_2$ with this equation yields $c_1\mathbf{u}_1^T\mathbf{u}_2 + c_2 = 0$. The two equations lead to

$$c_1\left(1 - \left|\mathbf{u}_1^T\mathbf{u}_2\right|^2\right) = 0.$$

Since $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly independent vectors $1 - \left|\mathbf{u}_1^T\mathbf{u}_2\right|^2 > 0$ (see (a) and (b) above), hence $c_1 = 0$. By the same token $c_2 = d_1 = d_2 = 0$.

Mark one and explain.

□ True □ False

2. (20) Let $\mathbf{v}_1,\ \mathbf{v}_2$ and $\mathbf{u}_1,\ \mathbf{u}_2$ be two pairs of linearly independent vectors. True or False? $\mathbf{v}_1\mathbf{u}_1^T + \mathbf{v}_2\mathbf{u}_2^T$ has rank 2.

Solution. First note that $\text{rank}AB \le \text{rank}B$ for any pair of matrices A and B when the product AB is defined. If $U = [\mathbf{u}_1, \mathbf{u}_2]$, and $V = [\mathbf{v}_1, \mathbf{v}_2]$, then $\mathbf{v}_1\mathbf{u}_1^T + \mathbf{v}_2\mathbf{u}_2^T = VU^T$, this shows that $\text{rank}\left(\mathbf{v}_1\mathbf{u}_1^T + \mathbf{v}_2\mathbf{u}_2^T\right) \le \text{rank}U^T = 2$.

Since $\mathbf{u}_1,\ \mathbf{u}_2$ are linearly independent vectors at least one coordinate of $\mathbf{u}_1$ is different from 0. If $\mathbf{u}_{1i} \ne 0$, then the i^{th} column of $\mathbf{v}_1\mathbf{u}_1^T + \mathbf{v}_2\mathbf{u}_2^T$, vector $u_{1i}\mathbf{v}_1 + u_{2i}\mathbf{v}_2 \ne 0$. Hence $\text{rank}\left(\mathbf{v}_1\mathbf{u}_1^T + \mathbf{v}_2\mathbf{u}_2^T\right) \ge 1$.

We next show that $\text{rank}\left(\mathbf{v}_1\mathbf{u}_1^T + \mathbf{v}_2\mathbf{u}_2^T\right) \ge 2$. Suppose the opposite, i.e.

$$\text{rank}\left(\mathbf{v}_1\mathbf{u}_1^T + \mathbf{v}_2\mathbf{u}_2^T\right) = 1.$$

Then for each pair $i \ne j$ the vectors

$$u_{1i}\mathbf{v}_1 + u_{2i}\mathbf{v}_2 \text{ and } u_{1j}\mathbf{v}_1 + u_{2j}\mathbf{v}_2$$

are linearly dependent, that is there are scalars c_1 and c_2, not all 0 so that

$$c_1\left(u_{1i}\mathbf{v}_1 + u_{2i}\mathbf{v}_2\right) + c_2\left(u_{1j}\mathbf{v}_1 + u_{2j}\mathbf{v}_2\right) = 0.$$

In other words

$$\left(c_1u_{1i} + c_2u_{1j}\right)\mathbf{v}_1 + \left(c_1u_{2i} + c_2u_{2j}\right)\mathbf{v}_2 = 0,$$

and

$$\det \begin{bmatrix} u_{1i} & u_{1j} \\ u_{2i} & u_{2j} \end{bmatrix} = 0.$$

Since the det is 0 for each pair of indecies i, j the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly dependent. This contradiction completes the proof.

Mark one and explain.

□ True □ False

3. (20) Let A be an $n \times n$ matrix so that $A^T A = I$. True or False? $\det A^2 = 1$.

Solution. Since $\det A^T = \det A$ one has $\det A^2 = \det A^T A = \det I = 1$.

Mark one and explain.

□ True □ False

4. (20) Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. If $\lambda_1 = 2$ and $\lambda_2 = 3$ are the eigenvalues of A compute $a_{11} + a_{22}$, and $\det A$.

Solution. $\operatorname{tr} A = a_{11} + a_{22} = \lambda_1 + \lambda_2 = 5$, $\det A = \lambda_1 \lambda_2 = 6$.

$a_{11} + a_{22} =$
$\det A =$

5. (20) Let $\mathbf{v},\ \mathbf{w} \in \mathbf{R}^n$, and $a = \mathbf{v}^T \mathbf{w}$. Consider an $n \times n$ matrix $A = \mathbf{v}\mathbf{w}^T$.

   (a) (10) Show that a and 0 are eigenvalues of A. Find an eigenvector $\mathbf{u}$ that corresponds to the eigenvalue a.

   Solution. If $\mathbf{u} = \mathbf{v}$, then $A\mathbf{u} = A\mathbf{v} = \mathbf{v}\mathbf{w}^T\mathbf{v} = a\mathbf{v} = a\mathbf{u}$.

   (b) (20) Find dimension $\dim V_a$ of the eigenspace that corresponds to the eigenvalue a, and dimension $\dim V_0$ of the eigenspace that corresponds to the eigenvalue 0.

   Solution.
   Case 1 Note that when $\mathbf{w} = 0$ the spaces V_a and V_0 are identical, and $V_a = V_0 = \mathbf{R}^n$.
   Case 2 We now assume that $\mathbf{w} \neq 0$. If $a = \mathbf{w}^T\mathbf{v} = 0$, then $V_a = V_0$. If $\mathbf{v} = 0$, then $V_a = V_0 = \mathbf{R}^n$. If $\mathbf{v} \neq 0$, then an eigenvector $\mathbf{u}$ should satisfy $0\mathbf{u} = A\mathbf{u} = \mathbf{v}(\mathbf{w}^T\mathbf{u})$, that is $\mathbf{u} \in \left\{\mathbf{w}^\perp\right\}$, and $\dim V_0 = \dim V_a = n - 1$.
   Case 3 Finally we focus on the case $\mathbf{w} \neq 0$, and $a = \mathbf{w}^T\mathbf{v} \neq 0$. If $\mathbf{u}$ is an eigenvector of A with the eigenvalue $a = \mathbf{w}^T\mathbf{v}$, then $0 \neq a\mathbf{u} = A\mathbf{u} = \mathbf{v}(\mathbf{w}^T\mathbf{u})$. This shows that $\mathbf{u} \in \operatorname{span}\{\mathbf{v}\}$, and $\dim V_a = 1$. On the other hand when $0 = A\mathbf{u} = \mathbf{v}(\mathbf{w}^T\mathbf{u})$ the dot product $\mathbf{w}^T\mathbf{u} = 0$, $\mathbf{u} \in \left\{\mathbf{w}^\perp\right\}$, and $\dim V_0 = n - 1$.