MATH411
quiz 1
03/24/20
Total 100
Solutions

By enrolling in this course, each student assumes the responsibilities of an active participant in UMBC's scholarly community in which everyone's academic work and behavior are held to the highest standards of honesty. Cheating, fabrication, plagiarism, and helping others to commit these acts are all forms of academic dishonesty, and they are wrong. Academic misconduct could result in disciplinary action that may include, but is not limited to, suspension or dismissal.

Show all work legibly. **Name**:______________________

1. (20) Let $\{p_1(z), \ldots, p_{n+1}(z)\}$ be elements of $\mathbf{P}_n$ so that $p_i(0) = 0$, $i = 1, \ldots, n+1$. True or False? The set $\{p_1(z), \ldots, p_{n+1}(z)\}$ is linearly independent.

   Solution. Note that $p_i(z) = zq_i(z)$, $q_i(z) \in \mathbf{P}_{n-1}$. Since $\{q_1(z), \ldots, q_{n+1}(z)\}$ are linearly dependent so are $\{p_1(z), \ldots, p_{n+1}(z)\}$.

   Mark one and explain.
   ▫ True ▫ False

2. (20) Let $W = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$, $n \geq 2$ be a vector set with at least one non zero vector. Denote by W_i a vector set with $n-1$ vectors obtained from W by removing $\mathbf{w}_i$. True or False? If $\mathbf{w}_i \in \text{span } W_i$ for each $i = 1, \ldots, n$, then $\dim \text{ span } W = 1$.

   Solution. Let $W = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4\}$ with $\mathbf{w}_1 = \mathbf{w}_2 = \mathbf{e}_1$, and $\mathbf{w}_3 = \mathbf{w}_4 = \mathbf{e}_2$. Note that $\dim \text{ span } W = 2$.

   Mark one and explain.
   ▫ True ▫ False

3. (60) Let $W = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$, $n \geq 2$ be a vector subset of a finite dimensional space V. If $i \in \{1, \ldots, n\}$ is the smallest index so that $\text{span } \{\mathbf{w}_1, \ldots, \mathbf{w}_{i-1}, \mathbf{w}_{i+1}, \ldots, \mathbf{w}_n\} = \text{span } W$, then denote $\{\mathbf{w}_1, \ldots, \mathbf{w}_{i-1}, \mathbf{w}_{i+1}, \ldots, \mathbf{w}_n\}$ by W^{-1}. If such an i does not exist, then define $W^{-1} = W$.

   (a) (15) True or False? If the vector set W is linearly independent, then $W^{-1} = W$.
   Solution. If the vector set W is linearly independent, then for each $i = 1, \ldots, n$ $\text{span } \{\mathbf{w}_1, \ldots, \mathbf{w}_{i-1}, \mathbf{w}_{i+1}, \ldots, \mathbf{w}_n\}$ is a proper subset of $\text{span } W$. Hence $W^{-1} = W$.
   Mark one and explain.
   ▫ True ▫ False

(b) (15) True or False? If $W^{-1} = W$, then the vector set W is linearly independent.

Solution. Let $c_1\mathbf{w}_1 + \ldots + c_n\mathbf{w}_n = 0$. If $c_1 \neq 0$, then $\mathbf{w}_1 \in \text{span}\,\{\mathbf{w}_2, \ldots, \mathbf{w}_n\}$, and $\text{span}\,\{\mathbf{w}_2, \ldots, \mathbf{w}_n\} = \text{span}\,W$. This contradicts $W^{-1} = W$. Repetition of this argument for $i = 2, \ldots, n$ completes the proof.

Mark one and explain.

□ True □ False

(c) (15) For $k > 1$ define W^{-k} as $\left(W^{-(k-1)}\right)^{-1}$. True or False? If $W^{-k} = W^{-(k+1)}$, then $W^{-k} = W^{-(k+m)}$ for each $m \geq 1$.

Solution. We shall use induction with respect to m. Note that the statement holds true for $m = 1$. Assume now that $W^{-k} = W^{-(k+m)}$ for $m > 1$. We show below that the statement holds true for $m + 1$.

$$W^{-(k+m+1)} = \left(W^{-(k+m)}\right)^{-1} = \left(W^{-k}\right)^{-1} = W^{-(k+1)} = W^{-k}.$$

Mark one and explain.

□ True □ False

(d) (15) True or False? The vector set $\bigcap_{i=1}^{\infty} W^{-i}$ is linearly independent.

Solution. Note that for each $k \geq 1$ one has $W^{-(k+1)} \subseteq W^{-k}$. This yields existence of $k \geq 1$ so that $\left(W^{-k}\right)^{-1} = W^{-(k+1)} = W^{-k}$. Hence due to (b) above W^{-k} is a linearly independent set, and $\bigcap_{i=1}^{\infty} W^{-i} = W^{-k}$.

Mark one and explain.

□ True □ False

4. (20) Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be linearly independent vectors. Suppose that $\mathbf{u}$ belongs to the span of each two vectors from $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. Find $\mathbf{u}$ as a linear combination of $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

Solution.

$$\begin{aligned} \mathbf{u} &= c_{31}\mathbf{u}_1 + c_{32}\mathbf{u}_2 && (1)\\ &= c_{21}\mathbf{u}_1 + c_{23}\mathbf{u}_3 && (2)\\ &= c_{12}\mathbf{u}_2 + c_{13}\mathbf{u}_3 && (3) \end{aligned}$$

Note that the difference between the right hand sides of the first and the second equations leads to

$$0 = [c_{31} - c_{21}]\mathbf{u}_1 + c_{32}\mathbf{u}_2 - c_{23}\mathbf{u}_3.$$

This yields

$$c_{31} = c_{21}, \text{ and } c_{32} = c_{23} = 0.$$

The difference between the right hand sides of the first and the last equations leads to

$$0 = c_{31}\mathbf{u}_1 + [c_{32} - c_{12}]\mathbf{u}_2 - c_{13}\mathbf{u}_3.$$

This yields

$$c_{32} = c_{12}, \text{ and } c_{31} = c_{13} = 0.$$

Since $c_{32} = 0$, and $c_{31} = 0$ one has $\mathbf{u} = 0$.

$\mathbf{u} =$