Name:

MATH225

Final Examination, 05/14/15

Total 100

Solutions

Show all work legibly.

1. (20) Consider the differential equation $y' + 4y - e^{-x} = 0$.

   (a) (5) Find a general solution $y_g(x)$ to the differential equation $y' + 4y = 0$.

$y_g(x) = ce^{-4x}$

(b) (10) Find a general solution $y_g(x)$ to the differential equation $y' + 4y = e^{-x}$.

$y_g(x) = ce^{-4x} + \frac{1}{3}e^{3x}$

(c) (5) Solve the inial value problem $y' + 4y - e^{-x} = 0,\ y(0) = \dfrac{4}{3}$.

The solution is $y(x) = e^{-4x} + \frac{1}{3}e^{3x}$

2. (60) Consider the differential equation $y'' + 4y' + 4y = \dfrac{e^{-2x}}{x^2}$, $x > 0$.

(a) (20) Find the general solution of $y'' + 4y' + 4y = 0$.

The solution is:

Solution. $\lambda^2+4\lambda+4 = 0$, hence $\lambda = -2$, and the general solution is $c_1 e^{-2x} + c_2 x e^{-2x}$.

(b) (20) Find a particular solution $y_p(x)$ of $y'' + 4y' + 4y = \dfrac{e^{-2x}}{x^2}$.

The solution is:

Solution. $y_p(x) = u_1(x)e^{-2x} + u_2(x)xe^{-2x}$, where

$$u_1(x) = -\int xe^{-2x}\frac{e^{-2x}}{x^2}\frac{1}{W(x)}\,dx,\ u_2(x) = \int e^{-2x}\frac{e^{-2x}}{x^2}\frac{1}{W(x)}\,dx,\ \text{where } W(x) = e^{-4x}.$$

Hence

$$u_1(x) = -\int \frac{1}{x}\,dx = -\ln x,\ \text{and } u_2(x) = \int \frac{1}{x^2}\,dx = -\frac{1}{x}.$$

$y_p(x) = -e^{-2x}\ln x - e^{-2x}$.

(c) (10) Find the general solution of $y'' + 4y' + 4y = \dfrac{e^{-2x}}{x^2}$, $x > 0$.

The solution is: $c_1 e^{-2x} + c_2 x e^{-2x} - e^{-2x} \ln x$.

(d) (10) Solve the initial value problem $y'' + 4y' + y = \dfrac{e^{-2x}}{x^2}$, $y(1) = 1$, $y'(1) = 0$.

The solution is:

Solution. The initial conditions yield:

$$1 = -c_1 - 2c_2, \text{ and } e^2 = c_1 + c_2.$$

Finally $c_1 = e^2$, $c_2 = -1 - e^2$, and $y(x) = e^2 e^{-2x} + [-1 - e^2]xe^{-2x} - e^{-2x} \ln x$.

3. (20) Consider the differential equation $y'' + y = 2^x$.

(a) (5) Find a general solution $y_g(x)$ to the differential equation $y'' + y = 0$.

$y_g(x) = c_1 \sin x + c_2 \cos x.$

(b) (15) Find a particular solution y_p to the differential equation $y'' + y = 2^x$.

$$y_p(x) = \frac{1}{1 + (\ln 2)^2} 2^x$$

(Hint: $2^x = e^{x \ln 2}$)

4. (10) Find the solution $y(t)$ for $y'' - y = 4\delta(t-2) + t^2,\ y(0) = 0,\ y'(0) = 2.$

$$Y = 4e^{-2s}\frac{1}{s^2-1} + 2\frac{s^2-s+1}{s^3(s-1)}$$

$$y(t) = 2\left(e^{t-2} + e^{-(t-2)}\right)u(t-2) - 2 - t^2 + 2e^t$$

5. (20) Compute $\int_{-\infty}^{\infty} \delta(x) \tan x^2 \, dx$.

$$\int_{-\infty}^{\infty} \delta(x) \tan x^2 \, dx = \tan 0 = 0.$$