MATH430/603

Final, 05/14/15

Total 100

Solutions

Show all work legibly.

1. (40) Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ and $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be two sets of linearly independent vectors in $\mathbf{R}^n$.

   (a) (20) True of False? For each i there is a vector $\mathbf{w}_i$ so that $\mathbf{v}_j^T \mathbf{w}_i = \delta_{ij}$.

   Solution. Let $\mathbf{w}_i \in \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_n\}$. Since $\mathbf{v}_i \notin \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_n\}$ one has $\mathbf{v}_i^T \mathbf{w}_i \neq 0$. Appropriate normalization of $\mathbf{w}_i$ completes the proof.

   Mark one and explain.

   □ True □ False

   (b) (20) True of False? The matrix $A = \sum_{i=1} \mathbf{u}_i \mathbf{v}_i^T$ is invertible.

   Solution. Let $\mathbf{w}_i \in \mathbf{R}^n$ such that $\mathbf{v}_j^T \mathbf{w}_i = \delta_{ij}$. Note that the vector set $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ is linearly independent, and $A\mathbf{w}_i = \mathbf{u}_i$.

   Mark one and explain.

   □ True □ False

2. (20) Let A be an $n \times n$ matrix such that dim $N(A) = n - 2$. True of False? There are vectors $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{v}_1$, $\mathbf{v}_2$ such that $A = \mathbf{u}_1 \mathbf{v}_1^T + \mathbf{u}_2 \mathbf{v}_2^T$.

   Solution. dim $N(A) = n - 2$ implies dim $R(A) = 2$. Suppose $A = [\mathbf{a}_1, \ldots, \mathbf{a}_n]$, and $R(A) = \text{span}\{\mathbf{a}_1, \mathbf{a}_2\}$. Note that $\mathbf{a}_i = c_{1i}\mathbf{a}_1 + c_{2i}\mathbf{a}_2$, $i = 3, \ldots, n$. We set $\mathbf{u}_1 = \mathbf{a}_1$, $\mathbf{u}_2 = \mathbf{a}_2$, $v_1 = \begin{bmatrix} 1 \\ 0 \\ c_{13} \\ \cdots \\ c_{1n} \end{bmatrix}$, and $v_2 = \begin{bmatrix} 0 \\ 1 \\ c_{23} \\ \cdots \\ c_{2n} \end{bmatrix}$, and the result follows.

   Mark one and explain.

   □ True □ False

3. (20) Let A and B be two $n \times n$ matrices such that $A = A^2$, $B = B^2$, and $AB = BA = 0$.

(a) Compute $C = \begin{bmatrix} A \\ B \end{bmatrix} [A + B][AB]$ (here $[AB]$ is an $n \times (2n)$ matrix with the first $n \times n$ block being A, and the second one B).

Solution.

$$\begin{bmatrix} A \\ B \end{bmatrix} [A + B][AB] = \begin{bmatrix} A \\ B \end{bmatrix} [AB] = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

(b) (20) True or False? rank $C =$ rank $A +$ rank B?

Solution. See (a) above.

Mark one and explain.

▫ True ▫ False

(c) (20) True or False? rank $\begin{bmatrix} A \\ B \end{bmatrix} [A\ B] = \text{rank}[A\ B]$?

Solution. Note that

$$\text{rank} \begin{bmatrix} A \\ B \end{bmatrix} [A\ B] = \text{rank}\ [A\ B] - \dim \left(N \begin{bmatrix} A \\ B \end{bmatrix} \bigcap R[A\ B] \right).$$

If $\mathbf{x} \in N \begin{bmatrix} A \\ B \end{bmatrix} \bigcap R[A\ B]$, then $0 = A\mathbf{x} = B\mathbf{x}$, and there are $\mathbf{y}_1$, $\mathbf{y}_2$ so that $\mathbf{x} = A\mathbf{y}_1 + B\mathbf{y}_2$. This yields

$$0 = A\mathbf{x} = A(A\mathbf{y}_1 + B\mathbf{y}_2) = A\mathbf{y}_1, \text{ and } 0 = B\mathbf{x} = B(A\mathbf{y}_1 + B\mathbf{y}_2) = B\mathbf{y}_2; \text{ i.e. } \mathbf{x} = 0.$$

Mark one and explain.

▫ True ▫ False

(d) (20) True or False? $\text{rank}[A\ B] = \text{rank}A + \text{rank}B$.

Solution. Let $A = [\mathbf{a}_1, \ldots, \mathbf{a}_k, \mathbf{a}_{k+1}, \ldots, \mathbf{a}_n]$, and $B[\mathbf{b}_1, \ldots, \mathbf{b}_l, \mathbf{b}_{l+1}, \ldots, \mathbf{b}_n]$ with first k and l columns of A and B respectively being linearly independent. Note that

$$c_1\mathbf{a}_1 + \ldots + c_k\mathbf{a}_k + d_1\mathbf{b}_1 + \ldots + d_l\mathbf{b}_l = 0$$

yields

$$0 = A(c_1\mathbf{a}_1 + \ldots + c_k\mathbf{a}_k + d_1\mathbf{b}_1 + \ldots + d_l\mathbf{b}_l) = c_1\mathbf{a}_1 + \ldots + c_k\mathbf{a}_k,$$

and

$$0 = B(c_1\mathbf{a}_1 + \ldots + c_k\mathbf{a}_k + d_1\mathbf{b}_1 + \ldots + d_l\mathbf{b}_l) = d_1\mathbf{b}_1 + \ldots + d_l\mathbf{b}_l.$$

Mark one and explain.

□ True □ False

(e) (20) True or False? $\begin{bmatrix} A \\ B \end{bmatrix} [A+B] = \begin{bmatrix} A \\ B \end{bmatrix}$.

Solution. Straightforward computation.

Mark one and explain.

□ True □ False

(f) (20) True or False? $\operatorname{rank} \begin{bmatrix} A \\ B \end{bmatrix} [A+B][AB] = \operatorname{rank} A + \operatorname{rank} B$.

Solution.

Mark one and explain.

□ True □ False

4. (20) Let $\mathbf{x} = \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} y_1 \\ \dots \\ y_n \end{bmatrix}$. Denote $\max\limits_i y_i$ by $\overline{y}$, and $\min\limits_i y_i$ by $\underline{y}$. True or False? If $\sum\limits_{i=1}^n x_i = 0$, then

$$\left|\mathbf{x}^T \mathbf{y}\right| \leq |\mathbf{x}|_1 \left(\overline{y} - \underline{y}\right).$$

Solution. Note that

$$x_1 y_1 + x_2 y_2 + \ldots + x_n y_n = -(x_2 + \ldots + x_n) y_1 + x_2 y_2 + \ldots + x_n y_n = x_2(y_2 - y_1) + x_3(y_3 - y_1) + \ldots + x_n(y_n - y_1)$$

Hence

$$\left|\mathbf{x}^T \mathbf{y}\right| \leq |x_2||y_2 - y_1| + \ldots + |x_n||y_n - y_1| \leq |x_2|(\overline{y} - \underline{y}) + \ldots + |x_n|(\overline{y} - \underline{y}) \leq |\mathbf{x}|_1 (\overline{y} - \underline{y}).$$

Mark one and explain.

□ True □ False