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Assignment 2:

1.
(a) Let $\mathbf{y}^T$ be represented component-wise as $[y_1, y_2, ..., y_m]$. Since $\mathbf{x}$ is an $n \times 1$ matrix, and $\mathbf{y}^T$ is a $1 \times m$ matrix, then $\mathbf{xy}^T$ is an $n \times m$ matrix, where

$$\text{column } 1 = y_1\,\mathbf{x},$$
$$\text{column } 2 = y_2\,\mathbf{x},$$
$$\vdots$$
$$\text{column } m = y_m\,\mathbf{x},$$

for $y_1, y_2, \ldots, y_m$: real scalars, and $\mathbf{x} \in \mathbf{R}^n$.

So, $m - 1$ columns can be written as a linear combination of a particular column times a scalar, e.g., in terms of an i^{th} column,

$$\text{column } 1 = (\text{column } i)(y_1/y_i),$$
$$\text{column } 2 = (\text{column } i)(y_2/y_i),$$
$$\vdots$$
$$\text{column } m = (\text{column } i)(y_m/y_i).$$

Here, column i is the only independent column in the matrix $\mathbf{xy}^T$.
Therefore, the column rank of $\mathbf{xy}^T$ is 1.

(b) True, $BB^T = A$.

Proof:

Since matrix B is an $m \times n$ array and B^T is an $n \times m$ array, then BB^T is an $m \times m$ array. Similarly for A, each $\mathbf{b_k b_k^T}$ term, where

$$\mathbf{b_k b_k}^T = [b_{k1}\ b_{k2} \ldots b_{km}]^T\ [b_{k1}\ b_{k2} \ldots b_{km}],$$

is an $m \times m$ array, whose sum for $1 \le k \le n$ is also an $m \times m$ array. Let the ij^{th} entry of BB^T be denoted by c_{ij}. Then, performing matrix multiplication yields,

$$c_{ij} = b_{i1}b_{1j} + b_{i2}b_{2j} + \ldots + b_{im}b_{mj}.$$

Let the ij^{th} entry of $\mathbf{b_k b_k}^T = d^{(k)}_{ij}$. Carrying out matrix multiplication gives $d^{(k)}_{ij} = b_{ik}b_{kj}$. Performing addition for $1 \le k \le n$ yields,

$$\Sigma_{1\le k\le n}\, d^{(k)}_{ij} = b_{i1}b_{1j} + b_{i2}b_{2j} + \ldots + b_{im}b_{mj}.$$

This sum is the ij^{th} entry of matrix A. Since $c_{ij} = \Sigma_{1\leq k\leq n}\, d^{(k)}{}_{ij}$ for the ij^{th} entries of BB^{T} and A respectively, therefore,

$$BB^{\text{T}} = A.$$

(c) If A is a symmetric matrix (incidentally, presuming the proof of 1(b) is correct, and A is indeed symmetric), then by the real spectral theorem, since A is a square, symmetric matrix, all eigenvalues of A are real.

2.

(a) Writing expression (1) from problem 2 in terms of $\mathbf{a}_1,\ldots, \mathbf{a}_m$, $\mathbf{x}$, and $\mathbf{y}$, yields

$$(1) = |\mathbf{a}_1 - (\mathbf{y} + (\mathbf{a}_1{}^{\text{T}}\mathbf{x})\ \mathbf{x})|^2 + |\mathbf{a}_2 - (\mathbf{y} + (\mathbf{a}_2{}^{\text{T}}\mathbf{x})\ \mathbf{x})|^2 + \ldots + |\mathbf{a}_m - (\mathbf{y} + (\mathbf{a}_m{}^{\text{T}}\mathbf{x})\ \mathbf{x})|^2.$$