Bruce James

Assignment 3:

1.

(a) From the associative properties of matrix multiplication,

$$\mathbf{x}^T\mathbf{B}\,\mathbf{B}^T\mathbf{x} = (\mathbf{x}^T\mathbf{B})(\mathbf{B}^T\mathbf{x}).$$

Since $\mathbf{x}^T\mathbf{B} = (\mathbf{B}^T\mathbf{x})^T$, and $\mathbf{x}^T\mathbf{B}$ is the transpose of a column vector $\mathbf{B}^T\mathbf{x}$, then $(\mathbf{x}^T\mathbf{B})(\mathbf{B}^T\mathbf{x})$ is a dot product of $\mathbf{B}^T\mathbf{x}$ with itself. From the positivity property of inner product spaces (i.e. $\langle v,v\rangle \geq 0$, for all $v \in \mathbf{R}^n$), then

$$\mathbf{x}^T\mathbf{B}\,\mathbf{B}^T\mathbf{x} \geq 0.$$

(b) True.

Reasoning:

Since $\mathbf{A}$ is a symmetric matrix, then by the (real) spectral theorem, $\mathbf{A}$ can be decomposed into the product $\mathbf{UDU}^T$, where $\mathbf{U}$ is an orthogonal matrix and $\mathbf{D}$ is diagonal, with real entries on the main diagonal equal to the eigenvalues of $\mathbf{A}$. Hence, the eigenvalues are real. *(From Elementary Linear Algebra, Ron Larson, 7th Ed., p.362; also http://web.mit.edu/jorloff/www/18.03-esg/notes/symmetricMatrices.pdf)*

(c) True.

Proof:

Since $\mathbf{A}$ is symmetric, then for any arbitrary eigenvalue λ of $\mathbf{A}$ satisfying $\mathbf{Ax} = \lambda\mathbf{x}$, $\lambda \in \mathbf{R}$, left multiplying $\mathbf{x}^T$ to both sides of $\mathbf{Ax} = \lambda\mathbf{x}$, yields

$$\mathbf{x}^T\mathbf{Ax} = \mathbf{x}^T\lambda\mathbf{x} = \lambda\mathbf{x}^T\mathbf{x}.$$

Since $\mathbf{A}$ is positive semi-definite,

$$\mathbf{x}^T\mathbf{Ax} \geq 0 \;\Rightarrow\; \lambda\mathbf{x}^T\mathbf{x} \geq 0,$$

and by positivity of inner product spaces,

$$\mathbf{x}^T\mathbf{x} \geq 0 \;\Rightarrow\; \lambda \geq 0.$$

Therefore, the eigenvalues of A are non-negative.

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2.

(a) Simplifying $|\mathbf{a} - \mathbf{x}\mathbf{a}^T\mathbf{x} - \mathbf{y}|^2$, with $\mathbf{x}^T\mathbf{x} = 1$, $\mathbf{x}^T\mathbf{y} = 0$:

Re-grouping as

$$
\begin{aligned}
|(\mathbf{a} - \mathbf{x}\mathbf{a}^T\mathbf{x}) - \mathbf{y}|^2 &= (\mathbf{a} - \mathbf{x}\mathbf{a}^T\mathbf{x})^2 - 2\mathbf{y}(\mathbf{a} - \mathbf{x}\mathbf{a}^T\mathbf{x}) + \mathbf{y}^2 \\
&= \mathbf{a}\cdot\mathbf{a} - 2\mathbf{a}(\mathbf{x}\mathbf{a}^T\mathbf{x}) + (\mathbf{x}\mathbf{a}^T\mathbf{x})^2 - 2\mathbf{a}\cdot\mathbf{y} + 2\mathbf{y}\cdot\mathbf{x}\mathbf{a}^T\mathbf{x} + \mathbf{y}\cdot\mathbf{y} \\
&= \mathbf{a}\cdot\mathbf{a} - 2(\mathbf{a}\cdot\mathbf{x})(\mathbf{a}\cdot\mathbf{x}) + (\mathbf{x}\cdot\mathbf{x})(\mathbf{a}^T\mathbf{x})^2 - 2\mathbf{a}\cdot\mathbf{y} + 2(\mathbf{y}\cdot\mathbf{x})\mathbf{a}^T\mathbf{x} + \mathbf{y}\cdot\mathbf{y} \\
&= \mathbf{a}\cdot\mathbf{a} - 2(\mathbf{a}^T\mathbf{x})^2 + (1)(\mathbf{a}^T\mathbf{x})^2 - 2\mathbf{a}\cdot\mathbf{y} + 2(\mathbf{0})\mathbf{a}^T\mathbf{x} + \mathbf{y}\cdot\mathbf{y} \\
&= \mathbf{a}\cdot\mathbf{a} - (\mathbf{a}^T\mathbf{x})^2 - 2\mathbf{a}^T\mathbf{y} + \mathbf{y}\cdot\mathbf{y} \\
&= (\mathbf{a} - \mathbf{y})^2 - (\mathbf{a}^T\mathbf{x})^2.
\end{aligned}
$$

So,

$$|\mathbf{a} - \mathbf{x}\mathbf{a}^T\mathbf{x} - \mathbf{y}|^2 = (\mathbf{a} - \mathbf{y})^2 - (\mathbf{a}^T\mathbf{x})^2.$$