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Assignment 6:

1. Since each $\mathbf{v}_i$ is a unit norm eigenvector, then when $i = j$,

$$\mathbf{v}_i^{\mathrm{T}}\mathbf{v}_j = \mathbf{v}_i^{\mathrm{T}}\mathbf{v}_i = ||\mathbf{v}_i||^2 = 1.$$

However, to show $\mathbf{v}_i^{\mathrm{T}}\mathbf{v}_j = 0$ when $i \neq j$, note that

$$M\mathbf{v}_i = \lambda_i\mathbf{v}_i \Rightarrow \mathbf{v}_i = (M\mathbf{v}_i)/\lambda_i\,, \quad (1)$$

and

$$M\mathbf{v}_j = \lambda_j\mathbf{v}_j \Rightarrow \mathbf{v}_j = (M\mathbf{v}_j)/\lambda_j\,, \quad (2)$$

for all $i, j \in \{1,2,\ldots, n\}$. Left multiplying the right-hand equations in (1) and (2) by $\mathbf{v}_j^{\mathrm{T}}$ and $\mathbf{v}_i^{\mathrm{T}}$ respectively, yields

$$\mathbf{v}_j^{\mathrm{T}}\mathbf{v}_i = (\mathbf{v}_j^{\mathrm{T}}M\mathbf{v}_i)/\lambda_i, \text{ and } \mathbf{v}_i^{\mathrm{T}}\mathbf{v}_j = (\mathbf{v}_i^{\mathrm{T}}M\mathbf{v}_j)/\lambda_j.$$

Taking the transpose of $\mathbf{v}_i^{\mathrm{T}}\mathbf{v}_j$ yields

$$[\mathbf{v}_i^{\mathrm{T}}\mathbf{v}_j]^{\mathrm{T}} = \mathbf{v}_j^{\mathrm{T}}\mathbf{v}_i \Rightarrow (\mathbf{v}_i^{\mathrm{T}}M\mathbf{v}_j)/\lambda_j = (\mathbf{v}_j^{\mathrm{T}}M\mathbf{v}_i)/\lambda_i. \quad (3)$$

Since $\mathbf{v}_i^{\mathrm{T}}\mathbf{v}_j = (\mathbf{v}_i^{\mathrm{T}}M\mathbf{v}_j)/\lambda_j$ is a scalar, it is equal to its transpose. So,

$$[\mathbf{v}_i^{\mathrm{T}}\mathbf{v}_j]^{\mathrm{T}} = \mathbf{v}_i^{\mathrm{T}}\mathbf{v}_j. \Rightarrow (\mathbf{v}_i^{\mathrm{T}}M\mathbf{v}_j)/\lambda_j = (\mathbf{v}_i^{\mathrm{T}}M\mathbf{v}_j)/\lambda_i. \quad (4)$$

From the premise that $i \neq j$ (and so $\lambda_i \neq \lambda_j$), and since $\mathbf{v}_i^{\mathrm{T}}M\mathbf{v}_j = \mathbf{v}_j^{\mathrm{T}}M\mathbf{v}_i$, then an order property of the real numbers implies that the right-hand equation in (4) is valid only when $(\mathbf{v}_i^{\mathrm{T}}M\mathbf{v}_j)/\lambda_j = (\mathbf{v}_j^{\mathrm{T}}M\mathbf{v}_i)/\lambda_i = 0$. So,

$$\mathbf{v}_i^{\mathrm{T}}\mathbf{v}_j = (\mathbf{v}_i^{\mathrm{T}}M\mathbf{v}_j)/\lambda_j = 0 \Rightarrow \mathbf{v}_i^{\mathrm{T}}\mathbf{v}_j = 0,$$

when $\lambda_i \neq \lambda_j$. Therefore $\mathbf{v}_i^{\mathrm{T}}\mathbf{v}_j = \delta_{ij}$.

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2. By contradiction, suppose that the set $\{\mathbf{v}_1, \mathbf{v}_2,\ldots, \mathbf{v}_n\}$ is linearly dependent. Then, at least one element of the set can be written as a linear combination of the remaining $n - 1$ elements, namely

$$\mathbf{v}_i = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_{i-1}\mathbf{v}_{i-1} + c_{i+1}\mathbf{v}_{i+1} + \ldots + c_n\mathbf{v}_n, \quad (5)$$

where at least one c_j in the set of constants is not equal to zero (in order to guarantee a non-trivial solution to the system).

Now, left multiplying (5) by $\mathbf{v}_i^{\mathrm{T}}$ yields

$$\begin{aligned}\mathbf{v}_i^{\mathrm{T}}\mathbf{v}_i &= \mathbf{v}_i^{\mathrm{T}}c_1\mathbf{v}_1 + \mathbf{v}_i^{\mathrm{T}}c_2\mathbf{v}_2 + \ldots + \mathbf{v}_i^{\mathrm{T}}c_{i-1}\mathbf{v}_{i-1} + \mathbf{v}_i^{\mathrm{T}}c_{i+1}\mathbf{v}_{i+1} + \ldots + \mathbf{v}_i^{\mathrm{T}}c_n\mathbf{v}_n \\ &= c_1\mathbf{v}_i^{\mathrm{T}}\mathbf{v}_1 + c_2\mathbf{v}_i^{\mathrm{T}}\mathbf{v}_2 + \ldots + c_{i-1}\mathbf{v}_i^{\mathrm{T}}\mathbf{v}_{i-1} + c_{i+1}\mathbf{v}_i^{\mathrm{T}}\mathbf{v}_{i+1} + \ldots + c_n\mathbf{v}_i^{\mathrm{T}}\mathbf{v}_n. \quad (6)\end{aligned}$$

From the results of problem 1 above, $\mathbf{v}_i^{\mathrm{T}}\mathbf{v}_i = 1$, and each term on the right of (6) is equal to zero, namely

$$c_1\mathbf{v}_i^{\mathrm{T}}\mathbf{v}_1 + c_2\mathbf{v}_i^{\mathrm{T}}\mathbf{v}_2 + \ldots + c_{i-1}\mathbf{v}_i^{\mathrm{T}}\mathbf{v}_{i-1} + c_{i+1}\mathbf{v}_i^{\mathrm{T}}\mathbf{v}_{i+1} + \ldots + c_n\mathbf{v}_i^{\mathrm{T}}\mathbf{v}_n = 0,$$

which implies the contradiction that 1 = 0. Therefore, no element of the mutually orthogonal set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ can be written as a linear combination of the remaining elements and, hence, is linearly independent.

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3. (a) Since $\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$, then left multiplying by M yields

$$\begin{aligned} M\mathbf{w} &= M(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n) \\ &= c_1 M\mathbf{v}_1 + c_2 M\mathbf{v}_2 + \ldots + c_n M\mathbf{v}_n \\ &= c_1 \lambda_1\mathbf{v}_1 + c_2 \lambda_2\mathbf{v}_2 + \ldots + c_n \lambda_n\mathbf{v}_n. \end{aligned}$$

Showing $M\mathbf{w} \neq \mathbf{0}$ by contradiction:

Suppose $M\mathbf{w} = \mathbf{0}$. Since $\mathbf{w}$ is a non-zero vector, then there exists at least one $c_i \neq 0$, $i \in \{1, 2, \ldots, n\}$. With $\lambda_i > 0$, and $\|\mathbf{v}_i\|^2 = 1$, then $c_i \lambda_i\mathbf{v}_i \neq \mathbf{0}$. With the contradiction assumption, then

$M\mathbf{w} = \mathbf{0} = c_1 \lambda_1\mathbf{v}_1 + c_2 \lambda_2\mathbf{v}_2 + \ldots + c_n \lambda_n\mathbf{v}_n.$

$\Rightarrow\ c_i \lambda_i\mathbf{v}_i = c_1 \lambda_1\mathbf{v}_1 + c_2 \lambda_2\mathbf{v}_2 + \ldots + c_{i-1}\lambda_{i-1}\mathbf{v}_{i-1} + c_{i+1}\lambda_{i+1}\mathbf{v}_{i+1} + \ldots + c_n \lambda_n\mathbf{v}_n.$

$\Rightarrow\ \mathbf{v}_i = (c_1 \lambda_1)/(c_i \lambda_i)\mathbf{v}_1 + (c_2 \lambda_2)/(c_i \lambda_i)\mathbf{v}_2 + \ldots + (c_{i-1}\lambda_{i-1})/(c_i \lambda_i)\mathbf{v}_{i-1}$
$\qquad + (c_{i+1}\lambda_{i+1})/(c_i \lambda_i)\mathbf{v}_{i+1} + \ldots + (c_n \lambda_n)/(c_i \lambda_i)\mathbf{v}_n.$

Therefore, $\mathbf{v}_i$ can be written as a linear combination of the remaining vectors in the mutually orthogonal set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, which is a contradiction.

Therefore, $M\mathbf{w} \neq \mathbf{0}$.

(b) With $M\mathbf{w} = c_1 \lambda_1\mathbf{v}_1 + c_2 \lambda_2\mathbf{v}_2 + \ldots + c_n \lambda_n\mathbf{v}_n$, left multiplying by M again, now yields

$$\begin{aligned} M^2\mathbf{w} &= M(c_1 \lambda_1\mathbf{v}_1 + c_2 \lambda_2\mathbf{v}_2 + \ldots + c_n \lambda_n\mathbf{v}_n) \\ &= c_1 \lambda_1 M\mathbf{v}_1 + c_2 \lambda_2 M\mathbf{v}_2 + \ldots + c_n \lambda_n M\mathbf{v}_n \\ &= c_1 \lambda_1^2\,\mathbf{v}_1 + c_2 \lambda_2^2\,\mathbf{v}_2 + \ldots + c_n \lambda_n^2\,\mathbf{v}_n. \end{aligned}$$

(c) After k iterations of left multiplication by M,

$$M^k\mathbf{w} = c_1 \lambda_1^k\,\mathbf{v}_1 + c_2 \lambda_2^k\,\mathbf{v}_2 + \ldots + c_n \lambda_n^k\,\mathbf{v}_n.$$

(d) With $M^k\mathbf{w} = c_1 \lambda_1^k \mathbf{v}_1 + c_2 \lambda_2^k \mathbf{v}_2 + \ldots + c_n \lambda_n^k \mathbf{v}_n$, then factoring out $c_1 \lambda_1^k$ yields

$$M^k\mathbf{w} = c_1 \lambda_1^k \left[\mathbf{v}_1 + (c_2/c_1)(\lambda_2^k/\lambda_1^k)\mathbf{v}_2 + \ldots + (c_n/c_1)(\lambda_n^k/\lambda_1^k)\mathbf{v}_n\right].$$

Also,

$$\|M^k\mathbf{w}\| = \left\|c_1 \lambda_1^k \left[\mathbf{v}_1 + (c_2/c_1)(\lambda_2^k/\lambda_1^k)\mathbf{v}_2 + \ldots + (c_n/c_1)(\lambda_n^k/\lambda_1^k)\mathbf{v}_n\right]\right\|$$

$$= |c_1 \lambda_1^k| \left\|\mathbf{v}_1 + (c_2/c_1)(\lambda_2^k/\lambda_1^k)\mathbf{v}_2 + \ldots + (c_n/c_1)(\lambda_n^k/\lambda_1^k)\mathbf{v}_n\right\|.$$

So,

$$M^k\mathbf{w} / \|M^k\mathbf{w}\| = \left[(c_1 \lambda_1^k)/ |c_1 \lambda_1^k|\right]\left[\left(\mathbf{v}_1 + (c_2/c_1)(\lambda_2^k/\lambda_1^k)\mathbf{v}_2 + \ldots\right.\right.$$

$$\left.+ (c_n/c_1)(\lambda_n^k/\lambda_1^k)\mathbf{v}_n\right)/\left(\left\|\mathbf{v}_1 + (c_2/c_1)(\lambda_2^k/\lambda_1^k)\mathbf{v}_2 + \ldots\right.\right.$$

$$\left.\left.+ (c_n/c_1)(\lambda_n^k/\lambda_1^k)\mathbf{v}_n\right\|\right)$$

As $k \to \infty$, then

$$\left\|\mathbf{v}_1 + (c_2/c_1)(\lambda_2^k/\lambda_1^k)\mathbf{v}_2 + \ldots + (c_n/c_1)(\lambda_n^k/\lambda_1^k)\mathbf{v}_n\right\| \to \|\mathbf{v}_1\| = 1,$$

and $(\lambda_i^k/\lambda_1^k) \to 0$, so

$$\left(\mathbf{v}_1 + (c_2/c_1)(\lambda_2^k/\lambda_1^k)\mathbf{v}_2 + \ldots + (c_n/c_1)(\lambda_n^k/\lambda_1^k)\mathbf{v}_n\right) \to \mathbf{v}_1.$$

The expression $(c_1 \lambda_1^k)/ |c_1 \lambda_1^k| = \pm 1$ as $k \to \infty$.

Therefore, $\lim_{k\to\infty} M^k\mathbf{w} / \|M^k\mathbf{w}\| = \pm\mathbf{v}_1$, when $c_1 \neq 0$.