Assignment 3

1. Assignment 2, Problem 1c.

   The matrix BB^T has a number of useful properties we will need shortly. For example, BB^T is symmetric (as has been shown by you), and also BB^T is positive semidefinite (i.e. for each vector $\mathbf{x}$ one has $\mathbf{x}^T BB^T\mathbf{x} \geq 0$). This remark leads to the following problems:

   (a) Let B be an $m \times n$ matrix and $\mathbf{x} \in \mathbf{R}^n$. Show that $\mathbf{x}^T BB^T\mathbf{x} \geq 0$.

   Next two problems deal with a general symmetric $n \times n$ matrix A:

   (b) True or False? If A is a symmetric matrix, then all eigenvalues of A are real. (This is the problem 1c from Assignment 2 that was left with no solution.)

   (c) True or False? If A is a symmetric and positive semidefinite matrix, then all its eigenvectors are nonnegative (i.e. if $\mathbf{x} \neq 0$, and $A\mathbf{x} = \lambda\mathbf{x}$, then $\lambda \geq 0$).

2. Assignment 2, Problem 2a.

   Ok, so you got the distance formula

$$\sum_{i=1}^{m} \left|\mathbf{a}_i - \mathbf{x}\mathbf{a}_i^T\mathbf{x} - \mathbf{y}\right|^2, \ \mathbf{x}^T\mathbf{x} = 1, \ \mathbf{x}^T\mathbf{y} = 0. \tag{1}$$

   Our goal is to identify vectors $\mathbf{x}$ and $\mathbf{y}$ so that (1) is minimized. If $f(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{m} \left|\mathbf{a}_i - \mathbf{x}\mathbf{a}_i^T\mathbf{x} - \mathbf{y}\right|^2$, then we have to deal with a constrained minimization problem

$$\min_{\mathbf{x},\mathbf{y}} \left\{f(\mathbf{x}, \mathbf{y}) \text{ subject to } \mathbf{x}^T\mathbf{x} = 1, \ \mathbf{x}^T\mathbf{y} = 0\right\}. \tag{2}$$

   Constrained minimization problems can be dealt with Lagrange multipliers (any memories from, perhaps, calculus?). However, before tackling the minimization problem (2) it would be good to simplify expression for f as much as possible. The expression for f is a sum of m squared distances, let's focus first on just one squared distance

$$\left|\mathbf{a} - \mathbf{x}\mathbf{a}^T\mathbf{x} - \mathbf{y}\right|^2.$$

   This is the dot product of the vector $\mathbf{a}-\mathbf{x}\mathbf{a}^T\mathbf{x}-\mathbf{y}$ with itself. Using the fact that $\mathbf{x}^T\mathbf{x} = 1$, and $\mathbf{x}^T\mathbf{y} = 0$ can the expression $\left(\mathbf{a} - \mathbf{x}\mathbf{a}^T\mathbf{x} - \mathbf{y}\right)^T \left(\mathbf{a} - \mathbf{x}\mathbf{a}^T\mathbf{x} - \mathbf{y}\right)$ be simplified? (perhaps grouping some terms together may help, for example one can write $\left(\mathbf{a} - \mathbf{x}\mathbf{a}^T\mathbf{x} - \mathbf{y}\right)$ as $\left([\mathbf{a} - \mathbf{y}] - \mathbf{x}\mathbf{a}^T\mathbf{x}\right)$, so that the dot product will eliminate some terms. perhaps other grouping might be useful). This leads to an open ended problem:

   (a) Simplify $\left|\mathbf{a} - \mathbf{x}\mathbf{a}^T\mathbf{x} - \mathbf{y}\right|^2$.