Assignment 5

1. General Comments.

- Assignment 4, Problem 1.

  I "buy" the solution. However I note that you use notations $\overline{A}$ for a matrix A, but do not explain it. It looks like $\overline{A}$ is obtained from A substituting entries a_{ij} of A by their conjugate $\overline{a}_{ij}$. Am I right?

- Assignment 4, Problem 1d.

  The conclusion $\overline{\lambda} = \lambda$ is based on the equality $\lambda \overline{\mathbf{x}}^T \mathbf{x} = \overline{\lambda} \overline{\mathbf{x}}^T \mathbf{x}$, or $(\lambda - \overline{\lambda}) \overline{\mathbf{x}}^T \mathbf{x} = 0$. The last equation indeed implies $\overline{\lambda} = \lambda$ **provided** $\overline{\mathbf{x}}^T \mathbf{x} \neq 0$. **Problem 5.1** Can you show that $\overline{\mathbf{x}}^T \mathbf{x} \neq 0$ when $\mathbf{x} \neq 0$?

- Assignment 4, Problem 2b iii.

  You claim $\mathbf{y} = \mathbf{c} - \frac{\mathbf{c}^T \mathbf{x}}{\mathbf{x}^T \mathbf{x}} \mathbf{x}$ where $\mathbf{c} = \frac{\mathbf{a}_1 + \ldots + \mathbf{a}_m}{m}$.
  Since we already agreed that $\mathbf{x}^T \mathbf{x} = 1$ the formula for $\mathbf{y}$ can be simplified

$$\mathbf{y} = \mathbf{c} - \mathbf{x} \left(\mathbf{c}^T \mathbf{x} \right). \tag{1}$$

- Assignment 4, Problem 2b i.

  You claim that $\mathbf{c} = \frac{\mathbf{a}_1 + \ldots + \mathbf{a}_m}{m}$ solves the minimization problem

$$\min_{\mathbf{z}} \sum_{i=1}^{m} |\mathbf{a}_i - \mathbf{z}|^2. \tag{2}$$

  Problem 5.2 Can you provide any proof?

- Centroid. Typically for a set of objects (usually numbers, or vectors) $\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ and a "distance" function $d(\mathbf{a}, \mathbf{b})$ a centroid $\mathbf{c}$ is defined as a solution of the minimization problem

$$\min_{\mathbf{x}} \sum_{i=1}^{m} d(\mathbf{x}, \mathbf{a}_i) \tag{3}$$

  (the solution does not have to be unique, hence I write "a centroid"). For example when $d(\mathbf{x}, \mathbf{a}_i) = |\mathbf{x} - \mathbf{a}_i|^2$ the centroid is the arithmetic mean. Here is a simple example of centroids very different from the arithmetic mean. Let $\{a_1, \ldots, a_m\}$ be a set of positive numbers, and

$$d(x, a_i) = a_i \left[\frac{x}{a_i} \log \frac{x}{a_i} - \frac{x}{a_i} + 1 \right]. \tag{4}$$

  Problem 5.3 For a set of m positive numbers $\{a_1, \ldots, a_m\}$ and the "distance" function provided by (4) find the centroid c.

- Assignment 4, Problem 2b ii.

  You claim that solution to the constrained minimization problem

$$\min_{\mathbf{z}} \left\{ \sum_{i=1}^{m} |\mathbf{a}_i - \mathbf{z}|^2, \ \mathbf{z}^T \mathbf{x} = 0 \right\}. \tag{5}$$

  Can be obtained as follows:

(a) Solve (2) and denote the solution by $\mathbf{c}$.

(b) Consider the the hyperplane $\{\mathbf{x}\}^{\perp} = \{\mathbf{w} \, : \, \mathbf{x}^T \mathbf{w} = 0\}$.

(c) Build $\mathbf{y}$–the projection of $\mathbf{c}$ on $\{\mathbf{x}\}^{\perp}$.

Problem 5.4 Can you prove that $\mathbf{y}$ constructed this way solves problem (5)?

2. New Problem.

   Formula (1) shows that the line we are looking for passes through the arithmetic mean of the vector set $\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$. Now we can plug $\mathbf{c} - \mathbf{x}\left(\mathbf{c}^T\mathbf{x}\right)$ for $\mathbf{y}$ in

$$\min_{\mathbf{x},\mathbf{y}} \left\{ \sum_{i=1}^{m} |\mathbf{a}_i - \mathbf{y}|^2 - \left|\mathbf{x}^T \mathbf{a}_i\right|^2 \text{ subject to } \mathbf{x}^T\mathbf{x} = 1, \ \mathbf{x}^T\mathbf{y} = 0 \right\}. \tag{6}$$

   (a) Plug $\mathbf{c} - \mathbf{x}\left(\mathbf{c}^T\mathbf{x}\right)$ for $\mathbf{y}$ in (6) and derive the resulting minimization problem.

   (b) Solve the minimization problem in (a) above.