Assignment 7, Part 2

1. Problem 7.3: If M is a symmetric $n \times n$ matrix, and $\lambda_1 > \lambda_2 > \ldots > \lambda_n$ are the real eigenvalues with the corresponding unit norm eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ (i.e. $M\mathbf{v}_i = \lambda_i \mathbf{v}_i$), then $\mathbf{v}_i^T \mathbf{v}_j = \delta_{ij}$ (that is $\mathbf{v}_i^T \mathbf{v}_j = 0$, if $i \neq j$, and $\mathbf{v}_i^T \mathbf{v}_i = 1$, and the vectors are "mutually orthogonal").

2. Problem 7.4: If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ are mutually orthogonal unit vectors, then the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly independent.

   Note that the first problem above deals with a pretty general situation, but not with the general one when some eigenvectors may be equal (the identity matrix I is the simplest example of a symmetric matrix with equal eigenvalues). We will take care of the equal eigenvalues case a bit later. The two problems above together show that the set of unit norm eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of a symmetric matrix forms a basis for the space (n linearly independent vectors in $\mathbf{R}^n$ form a basis). Here is the next problem.

3. Problem 7.5: Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be the unit norm eigenvectors of a symmetric matrix M with real eigenvalues $\lambda_1 > \lambda_2 > \ldots > \lambda_n > 0$. Consider a nonzero vector $\mathbf{w}$. This vector can be written as a linear combination of the basis elements

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n.$$

   (a) Compute $M\mathbf{w}$, and show that $M\mathbf{w} \neq 0$.

   (b) Compute $M^2\mathbf{w}$.

   (c) Compute $M^k\mathbf{w}$.

   (d) Compute $\lim_{k\to\infty} \dfrac{M^k\mathbf{w}}{|M^k\mathbf{w}|}$.