Assignment 4

1. Problem 1: prove that eigenvalues of a real symmetric matrix are real (Note: for this one I did eventually have to look online to get started with the complex conjugate proof idea)
   a. Since $A\boldsymbol{x} = \lambda\boldsymbol{x}$, for complex conjugates $\overline{A}\overline{\boldsymbol{x}} = \overline{\lambda}\overline{\boldsymbol{x}}$
   b. A is real, so $A = \overline{A}, and\ A\overline{\boldsymbol{x}} = \overline{\lambda}\overline{\boldsymbol{x}}$
   c. Multiply both sides of $A\boldsymbol{x} = \lambda\boldsymbol{x}$ by $\overline{\boldsymbol{x}}^T$, $\quad \overline{\boldsymbol{x}}^T A\boldsymbol{x} = \overline{\boldsymbol{x}}^T \lambda\boldsymbol{x}$
      i. $\overline{\boldsymbol{x}}^T \lambda\boldsymbol{x} = \overline{\boldsymbol{x}}^T A\boldsymbol{x} = (A^T\overline{\boldsymbol{x}})^T\boldsymbol{x} = (A\overline{\boldsymbol{x}})^T\boldsymbol{x} = \left(\overline{\lambda}\overline{\boldsymbol{x}}\right)^T\boldsymbol{x} = \overline{\boldsymbol{x}}^T\overline{\lambda}\boldsymbol{x}$
   d. So $\lambda = \overline{\lambda}$, and since the eigenvalue is equal to its complex conjugate it must be real.

2. Problem 2
   a. The expression I came up with is $\boldsymbol{a}^T\boldsymbol{a} - 2\boldsymbol{a}^T\boldsymbol{y} + \boldsymbol{y}^T\boldsymbol{y} - x^T\boldsymbol{a}\boldsymbol{a}^T x$
      i. $\boldsymbol{a}^T\boldsymbol{a} - 2\boldsymbol{a}^T\boldsymbol{y} + \boldsymbol{y}^T\boldsymbol{y} - x^T\boldsymbol{a}\boldsymbol{a}^T x = \boldsymbol{a}\cdot\boldsymbol{a} - \boldsymbol{a}\cdot\boldsymbol{y} - \boldsymbol{a}\cdot\boldsymbol{y} - \boldsymbol{y}\cdot\boldsymbol{y} - (\boldsymbol{x}^T\boldsymbol{a})^T\boldsymbol{x}^T\boldsymbol{a} = (\boldsymbol{a}-\boldsymbol{y})\cdot(\boldsymbol{a}-\boldsymbol{y}) - \boldsymbol{x}^T\boldsymbol{a}\cdot\boldsymbol{x}^T\boldsymbol{a} = |\boldsymbol{a}-\boldsymbol{y}|^2 - |\boldsymbol{x}^T\boldsymbol{a}|^2$

         So the expressions should be equal
   b. Find a $\boldsymbol{y}$ such that $\sum|\boldsymbol{a}_i - \boldsymbol{y}|^2$ from $i = 1$ to m, subject to $\boldsymbol{x}^T\boldsymbol{y} = 0$ is minimized.
      i. For a set of n dimentional vectors $\{\boldsymbol{a_1}, \boldsymbol{a_2}, \dots \boldsymbol{a_m}\}$, the centroid of the vectors is $\boldsymbol{c} = \frac{1}{m}(\boldsymbol{a}_1 + \boldsymbol{a}_2 + \cdots + \boldsymbol{a}_m)$. This should be the vector such that $\sum|\boldsymbol{a}_i - \boldsymbol{v}|^2$ from $i = 1$ to m is minimized if $\boldsymbol{v} = \boldsymbol{c}$.
      ii. Then the best $\boldsymbol{y}$ vector would be a vector orthogonal to $\boldsymbol{x}$ that gets closest to the centroid $\boldsymbol{c}$, which should be $\boldsymbol{c}$ minus its orthogonal projection onto $\boldsymbol{x}$
      iii. So $\boldsymbol{y} = \boldsymbol{c} - \frac{\boldsymbol{c}\cdot\boldsymbol{x}}{\boldsymbol{x}\cdot\boldsymbol{x}}\boldsymbol{x}$ where $\boldsymbol{c} = \frac{1}{m}(\boldsymbol{a}_1 + \boldsymbol{a}_2 + \cdots + \boldsymbol{a}_m)$ should minimize $\sum|\boldsymbol{a}_i - \boldsymbol{y}|^2$ from $i = 1$ to m, and $\boldsymbol{x}^T\boldsymbol{y} = 0$