Assignment 7

7.1 We know that BB^T is invertible, symmetric, and I'm assuming all the entries are real. We can use this to show that BB^T is positive definite. Let $\boldsymbol{z}$ be a nonzero n-dimentional vector. Then

$\mathbf{z}^T BB^T \mathbf{z} = (B^T\mathbf{z})^T(B^T\mathbf{z}) = |B^T\mathbf{z}|^2 \geq 0$, but since $\det(BB^T) \neq 0, |B^T\mathbf{z}|^2 > 0$, so BB^T is positive definite, positive definite matrices have positive eigenvalues.

7.2 Solve the linear system $(BB^T - \lambda I)\boldsymbol{v} = 0$ for $\boldsymbol{v}$. This can be done using the Gaussian elimination method.

7.3 Let λ_i and λ_j be two eigenvalues of the matrix BB^T, and let v_i and v_j be the corresponding eigenvectors. We will show the eigenvectors are orthogonal by showing their dot product is zero

Then we have

$$\lambda_i(v_i \cdot v_j) = \lambda_i v_i \cdot v_j = BB^T v_i \cdot v_j = (BB^T v_i)^T v_j = v_i^T (BB^T)^T v_j = v_i \cdot (BB^T)^T v_j =$$

$$v_i \cdot BB^T v_j = v_i \cdot \lambda_j v_j = \lambda_j (v_i \cdot v_j)$$

If eigenvalues are distinct, $\lambda_i \neq \lambda_j$, it must be the case that $(v_i \cdot v_j) = 0$ and so the eigenvectors of BB^T mutually orthogonal

7.4 Proof by contradiction. Let $\{v_1, v_2, \dots, v_n\}$ be a mutually orthogonal set of vectors. Assume that v_i is not linearly independent of the others. Then it can be written as a linear combination of the other vectors

$$v_i = c_1 v_1 + \cdots + c_{i-1}v_{i-1} + c_{i+1}v_{i+1} + \cdots + c_n v_n$$

Left multiply by v_i^T:

$$v_i^T v_i = c_1 v_i^T v_1 + \cdots + c_{i-1} v_i^T v_{i-1} + c_{i+1} v_i^T v_{i+1} + \cdots + c_n v_i^T v_n$$

$$v_i \cdot v_i = c_1(v_i \cdot v_1) + \cdots + c_{i-1}(v_i \cdot v_{i-1}) + c_{i+1}(v_i \cdot v_{i+1}) + \cdots + c_n(v_i \cdot v_n)$$

$$1 = 0$$

This is a contradiction, so the vectors must be linearly independent.

7.5 a) Let M be a symmetric matrix. M can be decomposed into ODO^T where O is an orthogonal matrix with the columns as the eigenvectors of M $\{v_1 \dots v_n\}$ and D is a diagonal matrix with the eigenvalues of M $\{\lambda_1 \dots \lambda_n\}$

$$M\boldsymbol{w} = ODO^T\boldsymbol{w} = OD\boldsymbol{w}',\ \boldsymbol{w}' = \begin{bmatrix} v_1 \cdot w \\ \vdots \\ v_n \cdot w \end{bmatrix} = \begin{bmatrix} v_1 \cdot (c_1 v_1 + \cdots + c_n v_n \\ \vdots \\ v_n \cdot (c_1 v_1 + \cdots + c_n v_n \end{bmatrix} =$$

$$\begin{bmatrix} c_1(v_1 \cdot v_1) + \cdots + c_n(v_1 \cdot v_n) \\ \vdots \\ c_1(v_n \cdot v_1) + \cdots + c_n(v_n \cdot v_n \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$ODw' = Ow'', w'' = \begin{bmatrix} \lambda_1 c_1 \\ \vdots \\ \lambda_n c_n \end{bmatrix}$

$Ow'' = \begin{bmatrix} \lambda_1 c_1 \boldsymbol{v}_1[1] + \lambda_2 c_2 \boldsymbol{v}_2[1] + \cdots + \lambda_n c_n \boldsymbol{v}_n[1] \\ \vdots \\ \lambda_1 c_1 \boldsymbol{v}_1[n] + \lambda_2 c_2 \boldsymbol{v}_2[n] + \cdots + \lambda_n c_n \boldsymbol{v}_n[n1] \end{bmatrix} = \lambda_1 c_1 \boldsymbol{v}_1 + \lambda_2 c_2 \boldsymbol{v}_2 + \cdots + \lambda_n c_n \boldsymbol{v}_n =$

$\sum \lambda_i c_i \boldsymbol{v}_i$ from $i = 1$ to n

This is nonzero because all the vectors are mutually orthogonal and they are simply scaled.

b) $M^2 \boldsymbol{w} = \sum \lambda_i^2 c_i \boldsymbol{v}_i$ from $i = 1$ to n

c) $M^k \boldsymbol{w} = \sum \lambda_i^k c_i \boldsymbol{v}_i$ from $i = 1$ to n

d) $\lim_{n\to\infty} \frac{M^k \boldsymbol{w}}{|M^k \boldsymbol{w}|} = \lim_{n\to\infty} \frac{\sum \lambda_i^k c_i \boldsymbol{v}_i}{|\sum \lambda_i^k c_i \boldsymbol{v}_i|} = \lim_{n\to\infty} \frac{\sum \lambda_i^k c_i \boldsymbol{v}_i}{\sqrt{\sum \lambda_i^{2k} c_i^2}}$ since the eigenvectors are orthogonal unit vectors

The limit of the sum is the sum of the limit

$$\lim_{n\to\infty} \frac{\sum \lambda_i^k c_i \boldsymbol{v}_i}{\sqrt{\sum \lambda_i^{2k} c_i^2}} = \sum \lim_{n\to\infty} \frac{\lambda_i^k c_i \boldsymbol{v}_i}{\sqrt{\sum \lambda_i^{2k} c_i^2}} = \sum \lim_{n\to\infty} \frac{\lambda_i^k c_i}{\sqrt{\sum \lambda_i^{2k} c_i^2}} \boldsymbol{v}_i = \sum \lim_{n\to\infty} \sqrt{\frac{\lambda_i^{2k} c_i^2}{\sum \lambda_i^{2k} c_i^2}} \boldsymbol{v}_i$$

Let λ_j be the maximum eigenvalue. Multiply each term in our summation by $\frac{1}{\lambda_j^{2k} c^2}$

For $i \neq j$: $\lim_{n\to\infty} \sqrt{\frac{\lambda_i^{2k} c_i^2}{\sum \lambda_i^{2k} c_i^2}} \boldsymbol{v}_i = 0$

For $i = j$: $\lim_{n\to\infty} \sqrt{\frac{\lambda_i^{2k} c_i^2}{\sum \lambda_i^{2k} c_i^2}} \boldsymbol{v}_i = \boldsymbol{v}_j$

And so the limit is the eigenvector corresponding to the largest eigenvalue.