ON GEVREY REGULARITY OF EQUATIONS OF FLUID AND GEOPHYSICAL FLUID DYNAMICS WITH APPLICATIONS TO 2D AND 3D TURBULENCE

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To my mother and father.

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The physical models of interest in this thesis are the Navier-Stokes equations (NSE) and surface quasi-geostrophic equation (SQG). We establish Gevrey regularity of solutions to these equations by combining Fourier analytic techniques with the semigroup approach of Weissler. This unifies several results regarding lower bound estimates on the radius of analyticity for the NSE, as well as provides an extension of the classical technique of Foias and Temam to so-called supercritical problems in the case of the SQG equation.

In the first part of this thesis, we analyze a general, subcritical system, which includes as special cases, the NSE and subcritical SQG equation. We show that in the case of the NSE, we recover the best-known estimates for the maximal radius of spatial analyticity for both the two-dimensional (2D) and three-dimensional (3D) NSE in the context of turbulence. Moreover, our results suggest a path for potential improvement in the 3D case.

The second part of the thesis is dedicated to the supercritical SQG equation. In this case, more care is needed when estimating the nonlinear term. In particular, the structure of the nonlinearity is exploited in a crucial way, in the form of a commutator, to ensure the Gevrey regularity of solutions. We present a method that extends the Gevrey norm technique of Foias and Temam to Besov spaces, as well as refines existing results concerning the regularity of solutions to the supercritical SQG equation in these spaces. We emphasize that the nature of Besov spaces and of the nonlinearity are exploited *together* in order to establish the desired estimates for the nonlinear term, for which we employ classical harmonic analysis techniques to derive.

CONTENTS

CHAPTER 1

INTRODUCTION

The equations believed to model viscous, incompressible fluid flow on a domain $\Omega \subset \mathbb{R}^2, \mathbb{R}^3$ are the Navier-Stokes equations (NSE):

$$\begin{cases} \partial_t u - \nu \Delta u + u \cdot \nabla u + \nabla p = f, \\ \nabla \cdot u = 0, \end{cases} \tag{1.1}$$

where $\nu > 0$ is the kinematic viscosity, u is the velocity vector field, p the scalar pressure field, and f a body force. The incompressibility condition is given by the relation $\nabla \cdot u = 0$, while $-\nu \Delta u$ represents the internal friction of the fluid. This system of equations is the starting point for the study of naturally occurring fluid motions such as turbulent motion and geophysical flows, e.g., movement of the mantle of the earth. Indeed, one can derive, for instance through scaling arguments and physical heursitics, other equations from the NSE such as the Boussinesq, shallow-water, and quasi-geostrophic equations. In particular, the quasi-geostrophic equation can be derived by placing (1.1) in a rotating coordinate frame such that the rotation is significant to the motion of the fluid, i.e., large Rossby number, and by assuming the pressure is balanced horizontally by the Coriolis force (geostrophic balance) and vertically by gravity (hydrostatic balance). The resulting boundary condition of such an equation yields the so-called surface quasi-geostrophic equation (SQG) over a domain $\Omega \subset \mathbb{R}^2$:

$$\begin{cases} \partial_t \theta + w_E \Lambda \theta + u \cdot \nabla \theta = 0, \\ u = \mathcal{R}^\perp \theta, \end{cases} \tag{1.2}$$

where $w_E > 0$ is a coefficient that comes from Ekman pumping at the boundary, θ is the fluid temperature, u is the velocity vector field, Λ is the Zygmund operator whose symbol is given by $\|\xi\|$, and $\mathcal{R}^\perp = (-R_2, R_1)$ is the perpendicular Riesz transform, where R_j is linear operator whose symbol is given by $\xi_j/\|\xi\|$. The fractionally dissipative versions of (1.1) and (1.2), i.e., with Λ^κ, $0 < \kappa \leq 2$, replacing Δ and Λ, are the main equations of study in this thesis. In particular, we study a specific type of higher-order regularity, called Gevrey regularity, which is a scale of regularity in between the classes C^∞, of smooth functions, and C^ω, of analytic functions. Our analysis of these equations will be organized into two groups: subcritical and critical/supercritical problems. By subcritical problems, we mean those equations for which the order of dissipation strictly dominates that of the nonlinearity, while critical/supercritical refer to those for which the opposite is true. With this language, we say that (1.1) is a subcritical problem, while (1.2) is a critical problem. Typically, subcritical problems are those for which perturbative methods can be applied in a straightforward manner, while critical and supercritical problems often require new ideas or for one to look to other methods.

NSE AND TURBULENCE

The study of Gevrey regularity of (1.1) was initiated by Foias and Temam in [40], where they pioneered a novel Gevrey norm approach to establish analyticity of solutions to the NSE in both space and time. An advantage of this approach is that it avoids having to make cumbersome recursive estimates on derivatives. Consequently, it has become a standard tool in estimating the analyticity radius for various equations (cf. [36, 70, 69, 66, 10, 8, 62, 64]). In the context of turbulence, the maximal or uniform radius of spatial analyticity, λ_a, has an important physical interpretation, namely, that it provides a lower bound for the so-called dissipation length scale, λ_d.

The conventional theory of turbulence posits the existence of certain universal length scales of paramount importance. For instance, according to Kolmogorov, there exists a *dissipation length scale*, λ_d, beyond which the viscous effects dominate the nonlinear coupling. This length scale can be characterized by the exponential decay of the energy density. Consequently, one expects the dissipation wave-number, $\kappa_d = \lambda_d^{-1}$, to majorize the inertial range where energy consumption is largely governed by the nonlinear effects and dissipation can be ignored. Since λ_a indicates a length scale beyond which the Fourier modes of the solution decay exponentially, one has by definition, $\lambda_d \gtrsim \lambda_a$. Much work, therefore, has been devoted towards studying the radius of analyticity of the Navier-Stokes equations.

Kolmogorov's theory for 3D turbulence asserts that

$$\lambda_d \sim \lambda_\varepsilon := (\nu^3/\varepsilon)^{1/4} , \tag{1.3}$$

where ν is viscosity and ε is the mean *energy dissipation rate* per unit mass. For 3D decaying turbulence, it was shown in [31] that

$$\lambda_a \sim \kappa_0^{-1}(\kappa_0\tilde{\lambda}_\varepsilon)^4 , \tag{1.4}$$

where $\tilde{\lambda}_\varepsilon$ is as in (1.3), except that the energy dissipation rate is a supremum in time rather than an averaged quantity. We can show that this estimate is valid for the *true* Kolmogorov length scale defined with the mean energy dissipation rate (see (2.61), (2.64)) under the

2/3-power law assumption (2.66) on the energy spectrum for a forced, turbulent flow, by means of an ensemble average with respect to an invariant measure (Theorem 8). It is valid on a large portion of the attractor (weak in the 3D case) the significance of which is quantified in terms of this measure. Ultimately, we can conclude that for any $0 < p < 1$,

$$\lambda_a \gtrsim_p \kappa_0^{-1}(\kappa_0 \lambda_\varepsilon)^4 \tag{1.5}$$

holds with probability $1 - p$ with respect to this invariant measure, where the suppressed constant tends to 0 as $p \to 1$. Similarly, a heuristic scaling argument by Kraichnan for 2D turbulence leads to

$$\lambda_d \sim \lambda_\eta := (\nu^3/\eta)^{1/6} \,, \tag{1.6}$$

where η is the mean *enstrophy dissipation rate* per unit mass (Theorem 10). We show that if the 2D power law (2.76) for the energy spectrum holds, then

$$\lambda_a \gtrsim_p \kappa_0^{-1}(\kappa_0 \lambda_\eta)^2 \tag{1.7}$$

holds with probability $1 - p$ with respect to some invariant measure.

These estimates actually follow from more general bounds on the radius of analyticity which require the solution to satisfy a certain "smallness" condition. Those conditions are met under the power law assumptions when averaged with respect to an invariant measure. Kukavica [61] achieved the same bound in 2D up to a logarithmic correction on all of the attractor using complex analytic techniques, interpolating between L^p norms of the initial data and the complexified solution, and invoking the theory of singular integrals. The approach in [61] was actually a modification of the approach in [50], where it was shown that $\lambda_d \gtrsim \nu(\sup_{t\leq T^*/2}\|u(t)\|_{L^\infty})^{-1}$. It is interesting to ask if these estimates can be obtained by working exclusively in frequency-space using Fourier techniques, rather than in physical space with the L^∞ norm. Indeed, this is an impetus of our work.

The technique presented here combines the use of Gevrey norms with the semigroup approach of Weissler [74] in ℓ^p spaces with $1 \leq p < \infty$. This norm and approach was applied in [11] to study spatial analyticity and Gevrey regularity of solutions to the NSE. However, the resulting estimate on the spatial radius of analyticity was not optimal for large data. This approach is refined here to obtain a sharper estimate for such data (Theorems 4, 5). In the case, $p = 1$, we work over a subspace of the Wiener algebra. The advantage of working in the Wiener algebra, $\mathcal{W}$, i.e. the Banach algebra of functions whose Fourier series converge absolutely, was explored in [70], where a sharp estimate on the radius of analyticity was obtained, for instance, for real steady states of the nonlinear Schrödinger equations. More recently, these ℓ^1-based Gevrey norms were also applied to the Szegö equation in [46] and the quasi-linear wave equation in [48]. In [46], an essentially sharp estimate on the radius is obtained there as well. While these works used energy-like approaches, the effectiveness and robustness of $\mathcal{W}$ as a working space to study analyticity has become increasingly clear. Indeed, the Wiener algebra is crucial to obtaining our estimate for the 2D NSE.

There are several advantages to our approach. First, our method is quite elementary. Since $\mathcal{W}$ is embedded in L^∞, we essentially recover the results of [50] and [61] without resorting to complex-analytic techniques and the theory of singular integrals, while furthermore allowing for rougher initial data. Secondly, by also working with phase spaces in ℓ^p for $1 < p < \infty$, we are able to unify the results of [31], [40], [50], and [61] . Thirdly, no logarithmic corrections appear in our estimates initially; they only appear when specializing to the context of 3D or 2D turbulence (see (2.80)). Finally, the method is rather robust and applies to a wide class of active and passive scalar equations with dissipation, including the quasigeostrophic (QG) equations.

THE SQG EQUATION

The SQG equation has received much attention over the years since it can be viewed as a toy model for the three-dimensional NSE and Euler equations. It is also of independent interest as it produces turbulent flows different from those arising from Navier-Stokes or Euler. For instance, the absence of anomalous dissipation in SQG turbulence has recently been established in [23], in contrast with three-dimensional turbulence where this phenomenon has been observed both numerically and experimentally.

The analytical and numerical study of the inviscid SQG equation ($w_E = 0$ case) was initiated by Constantin, Majda, and Tabak in [21], consequently sparking great interest within the mathematical community to study the SQG equation. In [24], Córdoba positively settled the conjecture from [21] that the formation of a simple type of blow-up could not occur. In general, however, formation of singularities for solutions of inviscid SQG is still open. Therefore, much focus has been directed towards studying (3.1) to explore the role of dissipation in preventing blow-up.

In the subcritical regime, the well-posedness of (3.1) was established by Resnick in [71], while the long-term behavior of its solutions were studied by Constantin and Wu in [18] and by Ju in [53]. Breakthrough in the critical case was met relatively recently in the papers of Caffarelli-Vasseur in [12] and Kiselev-Nazarov-Volberg in [59], where the problem of global regularity was settled by two very different methods. Since then, several different proofs of the global regularity problem have been discovered (cf. [16, 22, 32, 58]). From these techniques, global well-posedness for the critical case has also been established in other function spaces such as the Sobolev space $H^1(\mathbb{T}^2)$ in [22, 32], $H^1(\mathbb{R}^2)$ in [33], and the Besov space $B^{2/p}_{p,q}(\mathbb{R}^2)$ in [34]. The local well-posedness theory in critical spaces for the supercritical equations has been studied extensively as well (cf. [13, 14, 52, 68, 54, 75, 77]). These results have all been unified or extended by Chen-Miao-Zhang in [14] by working in the critical Besov spaces $B^{1+2/p-\kappa}_{p,q}(\mathbb{R}^2)$ (see (3.10) and (3.11)). In spite of these achievements, the global regularity problem for the *supercritical* case is still open. While this issue has been resolved in the "slightly" supercritical case in [27], where the dissipation is logarithmically enhanced, only conditional or so-called eventual regularity results are known (cf. [19, 26]).

Chapter 3 focuses on the supercritical case. In particular, we establish in Theorem **??** that the solutions to the initial value problem (3.1) with initial value θ_0 belonging to the critical Besov space, $B^{1+2/p-\kappa}_{p,q}(\mathbb{R}^2)$ immediately become Gevrey regular (see (3.16)) for at least a short time, and will remain Gevrey regular provided that the *homogeneous* Besov norm (see (3.12) and (3.13)) of the data is sufficiently small. Our result, therefore, properly extends that of Biswas in [9] to L^p-based Besov spaces and moreover, strengthens that of Dong and Li in [34], where it was shown that the solutions of Chen-Miao-Zhang in [14] are actually classical solutions. As a consequence of working with Gevrey norms, we obtain, as in [9], higher-order decay of the corresponding solutions (Corollary 31).

The study of Gevrey regularity or more generally, higher-order regularity of solutions to critical and subcritical SQG were previously pursued in ([8, 9, 32, 34, 35, 57]). The approach taken here is the one from [9], where it was shown that the solutions to critical and supercritical SQG with initial data belonging to the critical Sobolev space, $H^{2-\kappa}(\mathbb{R}^2)$, instantly become Gevrey regular.

One of the main issues in this thesis is how, due to supercriticality, the nonlinear term is estimated in a *Besov space-based* Gevrey norm (see (3.16)). It was observed by Miura in [68], for instance, that product estimates, in general, were insufficient to control the nonlinear term, thus motivating the use of commutators in order to take advantage of the cancellation inherent in the nonlinearity. We therefore view the nonlinear term as a bilinear multiplier operator (see (3.28)) arising from a certain commutator (see (3.31)) and obtain the corresponding $L^p \times L^q \to L^r$ bounds, where $1/r = 1/p + 1/q$ with $1 < p, r < \infty$ and $1 \leq q \leq \infty$ (see Theorem 32). This point of view was taken by Lemarié-Rieusset (cf. [65]) to prove spatial analyticity of solutions to the Navier-Stokes equations (NSE) starting from L^p initial data. His technique was successfully applied in the Besov space setting to the NSE in [4]. However, in the supercritical case, where one does not expect analyticity, the technique of Lemarié-Rieusset seems difficult to adapt. Nevertheless, one can obtain L^r bounds for a bilinear multiplier operator by establishing suitable decay estimates for derivatives of its symbol, from which one can then deduce boundedness. The celebrated Coifman-Meyer theorem comes to mind to accomplish this (cf. [15]). However, it is not applicable in our case (see (3.29)). Thus, we prove a multiplier theorem that accommodates our situation (see Theorem 35).

CHAPTER 2

A GENERAL SUBCRITICAL PROBLEM

2.1 PRELIMINARIES

Let $1 \leq k \leq n$, $1 \leq r < \alpha \leq 2$, and $L > 0$. We will consider the following general, subcritical initial value problem in $\Omega := [0, L]^n$:

$$\begin{cases} u_t + \nu_\alpha A^{\alpha/2} u + B_r(u, u) = f \\ u(x, 0) = u_0(x), \end{cases} \tag{2.1}$$

where ν_α has physical dimension length$^\alpha$/time, $u_0 : \Omega \to \mathbb{R}^k$ and $f : \Omega \times [0, T) \to \mathbb{R}^k$ are given, $u : \Omega \times [0, T) \to \mathbb{R}^k$ is unknown, A denotes the Laplacian with periodic boundary conditions, T is some linear operator, and $B_r = \kappa_0^{1-r} \widetilde{B}_r$, where $\widetilde{B}_r$ is *any* bilinear operator which satisfies

$$\left| \mathscr{F} \widetilde{B}_r(u, v)(k) \right| \lesssim |\kappa_0 k|^r (|\mathbf{u}| * |\mathbf{v}|)(k), \tag{2.2}$$

$\kappa_0 := 2\pi/L$, $\mathscr{F}$ denotes the Fourier transform, and $\mathbf{u}, \mathbf{v}$ denote the sequences $(\hat{u}(k))_{k \in \mathbb{Z}^n}$ and $(\hat{v}(k))_{k \in \mathbb{Z}^n}$, respectively. We assume that u_0, u, f are all L-periodic with mean-zero. In anticipation of our application (see Section 2.3), it will be convenient throughout to view u

as a *velocity*, which is to say that it has the physical dimension of length/time.

We will use the so-called wave-vector form of (2.1), which is simply (2.1) written in terms of its Fourier coefficients:

$$\begin{cases} \frac{d}{dt}\hat{u}(k,t) = -\nu_\alpha |\kappa_0 k|^\alpha \hat{u}(k,t) + \mathscr{F}B_r[\mathbf{u},\mathbf{u}](k,t) + \hat{f}(k,t), \\ \hat{u}(k,0) = \hat{u}_0(k), \end{cases} \tag{2.3}$$

where $\mathbf{u} \in (\mathbb{C}^n)^{\mathbb{Z}^n}$. Observe that (2.3) preserves the mean-zero condition, i.e., $\hat{u}(0,t) = 0$ for all $t > 0$. Consequently, we will work in the following sequence space as our ambient space:

$$\mathcal{K} := \{(\hat{u}(k)_{k\in\mathbb{Z}^n}) \in (\mathbb{C}^n)^{\mathbb{Z}^n} : \hat{u}(0) = 0, \hat{u}(k) = \hat{u}(-k)^*\}, \tag{2.4}$$

where $\hat{u}(k)^* := (\overline{\hat{u}_1(k)}, \ldots, \overline{\hat{u}_n(k)})$. Note that the condition $\hat{u}(k) = \hat{u}(-k)^*$ is simply that $\hat{u}(k) \in \mathbb{R}^n$. Now for $\sigma \in \mathbb{R}$ and $1 \le p \le \infty$ we define

$$V_{\sigma,p} := \{(\hat{u}(k))_{k\in\mathbb{Z}^n} \in (\mathbb{C}^n)^{\mathbb{Z}^n} : \|\mathbf{u}\|_{\sigma,p} < \infty\} \cap \mathcal{K}, \tag{2.5}$$

where

$$\|\mathbf{u}\|_{\sigma,p} := \left(\sum_{k\in\mathbb{Z}^n} |\kappa_0 k|^{\sigma p} |\hat{u}(k)|^p \right)^{1/p} \tag{2.6}$$

for $1 \le p < \infty$ and

$$\|\mathbf{u}\|_{\sigma,\infty} := \sup_{k\in\mathbb{Z}^n} |\kappa_0 k|^\sigma |\hat{u}(k)|. \tag{2.7}$$

For $\mathbf{u} \in \mathcal{K}$ we define the *Gevrey norm* of $\mathbf{u}$ by

$$\|\mathbf{u}\|_{\lambda,\sigma,p} := \left(\sum_{k \in \mathbb{Z}^n} e^{\lambda |\kappa_0 k| p} |\kappa_0 k|^{\sigma p} |\hat{u}(k)|^p \right)^{1/p} \tag{2.8}$$

if $1 \leq p < \infty$, and by

$$\|\mathbf{u}\|_{\lambda,\sigma,\infty} := \sup_{k \in \mathbb{Z}^n} e^{\lambda |\kappa_0 k|} |\kappa_0 k|^{\sigma} |\hat{u}(k)| \tag{2.9}$$

for $p = \infty$. We may then define the set $G_{\lambda,\sigma,p}$ by

$$G_{\lambda,\sigma,p} := \{\mathbf{u} \in (\mathbb{C}^n)^{\mathbb{Z}^n} : \|\mathbf{u}\|_{\lambda,\sigma,p} < \infty\}. \tag{2.10}$$

In other words, $G_{\lambda,\sigma,p} = e^{\lambda A^{1/2}} V_{\sigma,p}$ and $\|u\|_{\lambda,\sigma,p} = \|e^{\lambda A^{1/2}} u\|_{\sigma,p}$.

If $\mathbf{u}(\,\cdot\,)$ is time-dependent such that $\mathbf{u}(t) \in V_{\sigma,p}$, then we define the *Gevrey norm* of $\mathbf{u}(\,\cdot\,)$ by

$$\|\mathbf{u}(t)\|_{\lambda,\sigma,p} := \|\mathbf{u}(t)\|_{\lambda(t),\sigma,p} \tag{2.11}$$

for $\lambda := \lambda(t) : \mathbb{R}^+ \to \mathbb{R}^+$ increasing and sublinear, i.e., $\lambda(s+t) \leq \lambda(s) + \lambda(t)$ for all $s, t \geq 0$.

Observe that if $\mathbf{u} \in G_{\sigma,p}$, then the function u whose Fourier modes are represented by $\mathbf{u}$ satisfies the following higher-order decay estimates:

$$\|D^m u\|_{\sigma,p} \leq \left(\frac{m}{e}\right)^m (\kappa_0 \lambda)^{-m} \|\mathbf{u}\|_{\lambda,\sigma,p}, \tag{2.12}$$

for all $m \geq 0$. In fact, if a function has finite Gevrey norm, then the Fourier modes decay exponentially. Indeed, if $\|\mathbf{u}\|_{\lambda,\sigma} < \infty$, then

$$|\hat{u}(k)| \leq e^{-\lambda|k|}|k|^{-\sigma}\|\mathbf{u}\|_{\lambda,\sigma,p}. \tag{2.13}$$

This is in fact a characterization of analyticity. More precisely, we have the following proposition (cf. [66], [56]):

Proposition 1. Let $\sigma \in \mathbb{R}$ and $1 \leq p \leq \infty$.

1. If $\|\mathbf{u}\|_{\lambda,\sigma,p} < \infty$, then u admits an analytic extension on $\{x + iy : |y| < \lambda\}$;
2. If u has an analytic extension on $\{x + iy : |y| < \lambda\}$, then $\|\mathbf{u}\|_{\lambda',\sigma,p} < \infty$ for all $\lambda' < \lambda$.

Definition 1. If u is analytic, then we define the *maximal (uniform) radius of spatial analyticity* of u by

$$\lambda_{\max} := \sup\{\lambda' > 0 : \|\mathbf{u}\|_{\lambda',\sigma} < \infty\}. \tag{2.14}$$

Remark 2. For convenience, we adopt the following conventions for the rest of the chapter.

1. We will usually write $\mathbf{u}$ simply as u. It is convenient to view u as the function whose Fourier series have modes $\hat{u}(k)$, for $k \in \mathbb{Z}^n$.
2. By $u(t)$ or $u(k)$, or when the context is clear, simply u, we shall mean the time-dependent sequence $\mathbf{u}(t) = (\hat{u}(k,t))_{k \in \kappa_0 \mathbb{Z}^n}$, unless otherwise specified.
3. We will use $\lesssim$ to suppress extraneous absolute constants or physical parameters. In some instances, the dependence of these constants will be indicated as subscripts on $\lesssim$.
4. We will also use the notation $\sim$ to denote that the two-sided relation, $\lesssim$ and $\gtrsim$, holds.
5. Subscripts on constants will typically indicate the proposition they originate from, e.g., $C_\#$ is the constant from Lemma/Proposition/Theorem #.

6. Since constants will often depend on several parameters, we will often view constants as functions whose arguments are precisely these parameters.

Definition 2. Let $0 < T \leq \infty$, $\mathbf{u}_0 \in \mathcal{K}$, and $\mathbf{f} \in L^1(0, T; \mathcal{K})$. A *mild solution* of (2.1) is any $\mathbf{u} \in C([0, T]; \mathcal{K})$ such that

$$\mathbf{u}(t) = e^{-\nu_\alpha A^{\alpha/2}} u_0 + \int_0^t e^{-\nu_\alpha (t-s) A^{\alpha/2}} \mathbf{f}(s) \, ds - \int_0^t e^{-\nu_\alpha (t-s) A^{\alpha/2}} \mathbf{B}_r[\mathbf{u}, \mathbf{u}](s) \, ds, \qquad (2.15)$$

for all $0 \leq t \leq T$, where $\mathbf{B}_r$ denotes the sequence $(\mathscr{F} B_r(u, u)(k))_{k \in \mathbb{Z}^n}$, and satisfies

$$\int_0^t e^{-\nu_\alpha (t-s) |\kappa_0 k|^\alpha} |\mathscr{F} B_r[\mathbf{u}, \mathbf{u}](k, s)| \, ds < \infty \qquad (2.16)$$

for all $k \in \mathbb{Z}^n$.

Definition 3. Let $0 < T \leq \infty$, $\mathbf{u}_0 \in \mathcal{K}$, and $\mathbf{f} \in L^1(0, T; \mathcal{K})$. A *weak solution* of (2.1) is any $\mathbf{u} \in C([0, T]; \mathcal{K})$ such that $\mathbf{B}_r[\mathbf{u}, \mathbf{u}](k, t)$ exists for a.e. $t \in [0, T]$ and

$$\frac{d}{dt} \hat{u}(k, t) + \nu_\alpha |\kappa_0 k|^\alpha \hat{u}(k, t) + \mathscr{F} B_r[\mathbf{u}, \mathbf{u}](k, t) = \hat{f}(k, t) \qquad (2.17)$$

for all $k \in \mathbb{Z}^n$ and a.e. $t \in [0, T]$ and $\hat{u}(k, 0) = \hat{u}_0(k)$.

Definition 4. Let $0 < T \leq \infty$. A mild or weak solution $\mathbf{u}$ of (2.1) is *Gevrey regular* if there exists $1 \leq p \leq \infty$, $\sigma \in \mathbb{R}$, and $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ sublinear and increasing such that

$$\sup_{0 \leq t \leq T} \|\mathbf{u}(t)\|_{\lambda(t), \sigma, p} < \infty. \qquad (2.18)$$

SET-UP

For clarity, we will state our results in terms of scalar quantities, i.e., quantities with no physical dimensions. To this end, let $\omega_\alpha = \nu_\alpha \kappa_0^\alpha$ and observe that ω_α has physical dimensions of time^{-1}. For $1 \leq p, q \leq \infty$ and $0 < T_f \leq \infty$, we define

$$M_0 := \frac{\kappa_0^{-\sigma}}{\omega_\alpha \kappa_0^{-1}} \|u_0\|_{\sigma,p}, \tag{2.19}$$

$$M_f := \begin{cases} \frac{\kappa_0^{-\sigma}}{\omega_\alpha^2 \kappa_0^{-1}} \left(\omega_\alpha \int_0^{T_f} \|f(s)\|_{\lambda(s),\sigma,p}^q \, ds\right)^{1/q} & , 1 \leq q < \infty \\ \frac{\kappa_0^{-\sigma}}{\omega_\alpha^2 \kappa_0^{-1}} \sup_{0 \leq t \leq T_f} \|f(t)\|_{\lambda(t),\sigma,p} & , q = \infty \end{cases} \tag{2.20}$$

and

$$M := M_0 + M_f. \tag{2.21}$$

Then M_0, M_f, M are all scalar quantities.

To establish existence of solutions to (2.3), we will first establish existence of mild solutions, and then prove that such solutions are in fact weak solutions. For $0 < T \leq \infty$, $\sigma \in \mathbb{R}$, and $\beta \geq 0$, we will consider the spaces

$$X_T := \{u \in C([0,T]; V_{\sigma,p}) : \|u\|_X < \infty\}, \tag{2.22}$$

$$Y_T := \{u \in C((0,T]; V_{\sigma+\beta,p}) : \|u\|_Y < \infty\}, \tag{2.23}$$

$$Z_T := X_T \cap Y_T, \tag{2.24}$$

where X_T, Y_T, Z_T are equipped with the norms

$$\|u\|_X := \frac{\kappa_0^{-\sigma}}{\omega_\alpha \kappa_0^{-1}} \cdot \sup_{0 \leq t \leq T} \|u(t)\|_{\sqrt[\alpha]{\nu_\alpha t},\sigma}, \tag{2.25}$$

$$\|u\|_Y := \nu_\alpha^{\beta/\alpha} \frac{\kappa_0^{-\sigma}}{\omega_\alpha \kappa_0^{-1}} \cdot \sup_{0 < t \leq T} (t \wedge \omega_\alpha^{-1})^{\beta/\alpha} \|u(t)\|_{\sqrt[\alpha]{\nu_\alpha t},\sigma+\beta}, \tag{2.26}$$

$$\|u\|_Z := \max\{\|u\|_X, \|u\|_Y\}, \tag{2.27}$$

and $a \wedge b := \min\{a, b\}$. It is clear that X_T, Y_T, Z_T are Banach spaces with $Z_T \hookrightarrow X_T, Y_T$

continuously. Observe that these norms are scalar quantities as well.

The following abstract existence result provides conditions that ensure that one is in a perturbative regime. We note that this is a slightly generalized version of that found in [11]. We relegate its proof to Appendix A.

Theorem 3. Let $1 \leq p \leq \infty$ and $\sigma \in \mathbb{R}$. Let $Y, Z \subset C([0,T]; V_{\sigma,p})$ be Banach spaces with continuous embedding $i : Z \to Y$. Let $\Phi \in Z$ with $\|\Phi\|_Y \leq C_\Phi$ and define $E \subset Z$ by

$$E := \{u \in Z : \|u - \Phi\|_Z \leq C_\Phi\}. \tag{2.28}$$

Suppose $W = W(u, v)$ is given by

$$W(u,v)(t) := \int_0^t e^{-\nu_\alpha (t-s) A^{\alpha/2}} B[u(s), v(s)]\ ds, \tag{2.29}$$

for some bilinear function B, and satisfies, for some $N > 3(1 + \|i\|_{Z\to Y})$

$$\|W(u,\ \cdot\ )\|_{Y\to Z}, \|W(\ \cdot\ , u)\|_{Y\to Z} \leq \frac{1}{N}, \tag{2.30}$$

whenever $u \in E$. Then there exists a unique $u \in E$ such that

$$u = \Phi - W(u,u) \tag{2.31}$$

□

Indeed, by the Duhamel principle, the solution u that we seek will be a fixed point of the operator S defined by

$$(Su(\,\cdot\,))(t) := \underbrace{e^{-\nu_\alpha t A^{\alpha/2}} u_0 + \int_0^t e^{-\nu_\alpha (t-s) A^{\alpha/2}} f(s)\, ds}_{\Phi(t)} - \underbrace{\int_0^t e^{-\nu_\alpha (t-s) A^{\alpha/2}} B_r[u(s), u(s)]\, ds}_{W(u,u)(t)}\,. \tag{2.32}$$

In particular, we establish the existence of such a function u in the closed subset $E_T \subset Z_T$ for some $T > 0$, where E_T is defined by (2.28) for some $C > 0$ with $\|\Phi\|_Y \leq C$.

2.2 MAIN RESULTS

Our first main theorem guarantees existence of Gevrey regular weak solutions to (2.3) provided that f is analytic and $u_0 \in V_{\sigma,p}$ and gives an improved estimate on the corrresponding maximal radius of spatial analyticity.

Theorem 4. Let $n \geq 1$, $1 < p < \infty$, and $\sigma \in \mathbb{R}$ be given such that they satisfy

$$\frac{n}{p'} - (\alpha - r) < \sigma < \frac{n}{p'}, \tag{2.33}$$

where p, p' are Hölder conjugates. Suppose $u_0 \in V_{\sigma,p}$ and $e^{\sqrt[\alpha]{\nu_\alpha}\,\cdot A^{1/2}} f \in L^q(0, T_f; V_{\sigma,p})$ for some $1 < q \leq \infty$, where $0 < T_f \leq \infty$. Then there exists a time $0 < T^* < \infty$ and a mild solution $u \in C([0, T^*]; V_{\sigma,p})$ to (2.1) such that u is a Gevrey regular weak solution whose maximal radius of spatial analyticity at time T^* satisfies

$$\lambda_a(T^*) \geq C^* \kappa_0^{-1} M^{-\frac{1}{(\alpha - r) - n/p' + \sigma}}, \tag{2.34}$$

for some $C^* := C^*(n, p, q, r, \alpha, \beta, \sigma)$, where M is given by (2.20). □

The $p = 1$ version is similar, except that it allows for the smoothness index, σ, to be negative.

Theorem 5. Let $1 < q \leq \infty$ and q, q' be Hölder conjugates. Suppose $\sigma, \beta \in \mathbb{R}$ satisfy $\sigma_- \leq \beta < \min\{r, \alpha - r, \alpha/2, \alpha/q'\}$, where $\sigma_- := \max\{0, -\sigma\}$. Let $u_0 \in V_{\sigma,1}$ and $e^{\sqrt[\alpha]{\nu_\alpha} \cdot A^{1/2}} f \in L^q(0, T_f; V_{\sigma,1})$, where $0 < T_f \leq \infty$. Then there exists a time $0 < T^* \leq T_f$ and a mild solution $u \in C([0, T^*]; V_{\sigma,1})$ to (2.1) such that u is a Gevrey regular weak solution whose maximal radius of analyticity at time T^* satisfies

$$\lambda_a(T^*) \geq C^* \kappa_0^{-1} M^{-\frac{1}{(\alpha - r) - \beta}}, \tag{2.35}$$

for some $C^* = C^*(r, \alpha, \beta, \sigma)$. □

The next two theorems show that if the initial value satisfies certain upper bounds, then the corresponding maximal radius of spatial analyticity satisfies sharper lower bounds.

Theorem 6. Let $n \geq 1$, $1 < p, p' < \infty$, and $\sigma \in \mathbb{R}$ be given such that they satisfy

$$\frac{n}{p'} - (\alpha - r) < \sigma < \frac{n}{p'}, \tag{2.36}$$

where p, p' are Hölder conjugates. Suppose that $u_0 \in V_{\sigma,p}$ and $e^{\sqrt[\alpha]{\nu_\alpha} \cdot A^{1/2}} f \in L^q(0, T_f; V_{\sigma,p})$, for some $1 < q \leq \infty$, satisfy

$$M_0 \leq C_* M_f^{\frac{(\alpha - r) - n/p' + \sigma}{(\alpha - r) - n/p' + \sigma + \alpha/q'}}, \tag{2.37}$$

for some $C_* > 0$, where q, q' are Hölder conjugates, and M_f is given by (2.20). Then there exists a time $0 < T^* < \infty$ and a mild solution $u \in C([0, T^*]; V_{\sigma,p})$ to (2.1) such that u is a Gevrey regular weak solution whose maximal radius of spatial analyticity at time T^*

satisfies

$$\lambda_a(T^*) \geq \kappa_0^{-1} \begin{cases} 1, & M_f \leq (C^*)^{\alpha-r-n/p'+\sigma+\alpha/q'} \\ C^* M_f^{-\frac{1}{\alpha-r-n/p'+\sigma+\alpha/q'}}, & M_f > (C^*)^{\alpha-r-n/p'+\sigma+\alpha/q'} \end{cases} \tag{2.38}$$

for some $C^* := C^*(n, p, q, r, \alpha, \beta, \sigma)$.

Again, a corresponding result for the $p = 1$ case also holds.

Theorem 7. Let $1 < q \leq \infty$ and q, q' be Hölder conjugates. Suppose that $\sigma, \beta \in \mathbb{R}$ satisfy $\sigma_- \leq \beta < \min\{r, \alpha - r, \alpha/2, \alpha/q'\}$, where $\sigma_- := \max\{0, -\sigma\}$. Suppose that $u_0 \in V_{\sigma,1}$ and $e^{\sqrt[\alpha]{\nu_\alpha} \cdot A^{1/2}} f \in L^q(0, T_f; V_{\sigma,1})$, for some $1 < q \leq \infty$, satisfy

$$M_0 \leq C_* M_f^{\frac{(\alpha-r)-\beta}{(\alpha-r)-\beta+\alpha/q'}}, \tag{2.39}$$

for some $C_* > 0$, then there exists $T^* < T_f$ and mild solution $u \in C([0, T^*]; V_\sigma)$ to (2.3) such that u is also a Gevrey regular weak solution, with radius of analyticity at time T^* satisfying

$$\lambda_a(T^*) \geq \kappa_0^{-1} \begin{cases} 1, & M_f \leq (C^*)^{(\alpha-r)-\beta+\alpha/q'} \\ C^* M_f^{-\frac{1}{(\alpha-r)-\beta+\alpha/q'}}, & M_f > (C^*)^{\alpha-r-\beta+\alpha/q'} \end{cases} \tag{2.40}$$

for some $C^* := C^*(q, r, \alpha, \beta, \sigma)$.

In the next section, we discuss some applications of Theorems 4 and 6.

2.3 APPLICATIONS

2.3.1 SPECIAL CASES OF (2.1)

The system (2.1) actually contains the Navier-Stokes and subcritical surface quasi-geostrophic equations as special cases.

NAVIER-STOKES EQUATIONS

Recall that the Navier-Stokes equations are given by

$$\begin{cases} \partial_t u - \nu \Delta u + u\cdot \nabla u + \nabla p = F, \\ \nabla\cdot u = 0, \\ u(x,0) = u_0(x). \end{cases} \tag{2.41}$$

with periodic boundary conditions, where u, u_0, p, F are all L-periodic with zero mean. One can eliminate the pressure by applying the Helmholtz-Leray orthogonal projection, $\mathcal{P}$, i.e., projection onto divergence-free vector fields:

$$\mathcal{P}(\hat{u}(k)e^{i\kappa_0 k\cdot x}) = \left(\hat{u}(k) - \left(\frac{k}{|k|}\cdot \hat{u}(k)\right)\frac{k}{|k|}\right)e^{i\kappa_0 k\cdot x}, \quad (k \in \mathbb{Z}^n). \tag{2.42}$$

Using the divergence-free condition, one then arrives at

$$\begin{cases} \partial_t u - \nu \Delta u + B(u,u) = f, \\ \nabla\cdot u = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{2.43}$$

where $f = \mathcal{P}F$ and $B(u,u) = \mathcal{P}\sum_{j=1}^{n} \partial_j(u_j u)$. Observe that by (2.42) B satisfies

$$|\mathscr{F}B(u,v)(k)| \leq 2\sum_{j=1}^{n}\sum_{\ell\in\mathbb{Z}^n} k_j \hat{u}_j(k-\ell)\hat{v}(\ell) \leq 2|k||\mathbf{u}| * |\mathbf{v}|)(k) \tag{2.44}$$

for all $k \in \mathbb{Z}^n$, where $\mathbf{u}$ and $\mathbf{v}$ denote the sequences $(\hat{u}(k))_{k\in\mathbb{Z}^n}$ and $(\hat{v}(k))_{k\in\mathbb{Z}^n}$, respectively. Thus, (2.1) reduces to (2.43) when $k = n$, $r = 1$, and $B_r = B$. In this case, one needs to include the divergence-free condition into the space $\mathcal{K}$ defined in (2.45). In particular, one should replace $\mathcal{K}$ by $\mathcal{K}_0$:

$$\mathcal{K}_0 := \{(\hat{u}(k)_{k\in\mathbb{Z}^n}) \in (\mathbb{C}^n)^{\mathbb{Z}^n} : \hat{u}(0) = 0, \hat{u}(k) = \hat{u}(-k)^*, k\cdot \hat{u}(k) = 0\}, \tag{2.45}$$

One can also eliminate the pressure by taking the curl of (2.41). The resulting system is the so-called vorticity formulation of (2.41). In three-dimensions we have

$$\begin{cases} \partial_t\omega - \nu\Delta\omega + u\cdot\nabla\omega + \omega\cdot\nabla u = f, \\ \nabla\cdot\ u = 0, \\ \omega = \nabla\times u, \\ \omega(x,0) = \nabla\times u_0(x), \end{cases} \tag{2.46}$$

where this time $f = \nabla\times F$. We recall that u can be recovered from ω through the Biot-Savart law, i.e.,

$$u(x) = p.v.\int \frac{x-y}{|x-y|^3}\times\omega(y)\ dy. \tag{2.47}$$

Indeed, a direct computation shows that

$$-\Delta u = \begin{pmatrix} \partial_2\omega_3 - \partial_3\omega_2 \\ \partial_3\omega_1 - \partial_1\omega_3 \\ \partial_1\omega_2 - \partial_2\omega_1 \end{pmatrix}. \tag{2.48}$$

Thus, $u = T\omega$ for some singular integral operator T. Since $\nabla\cdot\omega = 0$ always holds, we can rewrite (2.46) as

$$\begin{cases} \partial_t\omega - \nu\Delta\omega + B(\omega,\omega) + = f, \\ \nabla\cdot\ (T\omega) = 0, \\ \omega = \nabla\times u, \\ \omega(x,0) = \nabla\times u_0(x), \end{cases} \tag{2.49}$$

where B is defined by $B(u,v) = (Tu)\cdot\nabla v + u\cdot\nabla(Tv)$ and ω, ω_0, f are L-periodic with zero mean. Now observe that (2.48) implies that

$$|\widehat{T\omega}(k)| \leq \frac{2}{|\kappa_0 k|}|\hat{\omega}(k)| \leq 2|\hat{\omega}(k)| \tag{2.50}$$

for all $k \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$. Thus, by (2.50), if u, v are divergence-free and have zero mean, then

$$|\mathscr{F}B(u,v)(k)| \leq 2|k|(|\mathbf{u}| * |\mathbf{v}|)(k) \tag{2.51}$$

and, as before, (2.1) reduces to (2.46) with $k = n = 3$, $r = 1$, and $B_r = B$.

Similarly, in two-dimensions we have

$$\begin{cases} \partial_t \omega - \nu \Delta \omega + (T\omega)\cdot \nabla \omega = f, \\ \nabla \cdot \ u = 0, \\ \omega = \nabla \times u, \\ \omega(x,0) = \nabla \times u_0(x), \end{cases} \tag{2.52}$$

which can be rewritten as (2.49), except with B given by $B(u,v) = (Tu)\cdot \nabla v$ and T by

$$T\omega := -(-\Delta)^{-1} \nabla^{\perp} \omega, \tag{2.53}$$

where $\nabla^{\perp} = (-\partial_2, \partial_1)$. In fact, in this case we have

$$|\widehat{T\omega}(k)| \leq \frac{1}{|\kappa_0 k|} |\hat{\omega}(k)|. \tag{2.54}$$

Hence, (2.1) also reduces to the two-dimensional vorticity formulation with $k = n = 2$, $r = 1$, and $B_r = B$.

SUBCRITICAL QUASI-GEOSTROPHIC EQUATION

Now let $1 < \alpha \leq 2$. We recall that the forced subcritical quasi-geostrophic equation is given by

$$\begin{cases} \partial_t \theta + \kappa \Lambda^{\alpha} \theta + u \cdot \nabla \theta \quad = f \\ u = \mathcal{R}^{\perp} \theta \\ \theta(x,0) = \theta_0(x), \end{cases} \tag{2.55}$$

where Λ is the Zygmund operator, $\mathscr{F}(\Lambda\theta)(k) = |k|\hat{\theta}(k)$, $\mathcal{R}^\perp = (-R_2, R_1)$, where R_j is the j-th Riesz transform, which is defined by $\mathscr{F}(R_j\theta)(k) = -(k_j/|k|)\hat{\theta}(k)$ for $k \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$, $\theta : \Omega \to \mathbb{R}$ represents temperature, and $u : \Omega \to \mathbb{R}^2$ is velocity. We suppose, as before, that θ_0, f are L-periodic with zero mean. An elementary computation shows that the solution θ must also have zero mean. In particular, we have

$$\begin{cases} \partial_t\theta + \kappa\Lambda^\alpha\theta + B(\theta,\theta) = f, \\ \theta(x,0) = \theta_0(x), \end{cases} \tag{2.56}$$

where $B(u,v) = (Tu)\cdot \nabla v$ and $Tu = \mathcal{R}^\perp u$. Observe that

$$\mathscr{F}(\nabla\cdot (Tu))(k) = k_1(k_2/|k|)\hat{u}(k) - k_2(k_1/|k|)\hat{u}(k) = 0. \tag{2.57}$$

Thus, Tu is divergence-free. Since R_j is a Calderón-Zygmund operator we again have

$$|\mathscr{F}B(u,v)(k)| \le C|k|(|\mathbf{u}| * |\mathbf{v}|)(k). \tag{2.58}$$

Therefore, (2.1) reduces to (2.55) when $k = 1, n = 2$, $r = 1$, and $B_r = B$.

2.3.2 APPLICATION TO TURBULENT FLOWS

In this subsection, we show how our results in Theorems 4, 6 improve the known estimates for λ_d for turbulent flows. While their "smallness" assumptions may not hold on all of the 2D global (3D weak) attractor, in the context of turbulence, one can expect these conditions to hold *on average*, in a precise sense.

The statistical theory of turbulence concerns relations between quantities that are averaged, either with respect to time or over an ensemble of flows, e.g. results from repeated experiments. It is remarkable that these two seemingly different approaches are in fact related.

The mathematical equivalent of a large time average is rigorously expressed in terms of Banach limits. Following [42], define the space H by

$$H := \{(\hat{u}(k))_{k\in\mathbb{Z}^n} \in (\mathbb{C}^n)^{\mathbb{Z}^n} : \|\mathbf{u}\|_{\ell^2} < \infty\} \cap \mathcal{K}. \tag{2.59}$$

Let Φ be a real-valued weakly continuous function on H. Then for any weak solution u of (2.3) on $[0,\infty)$, there exists a probability measure μ for which

$$\langle\Phi\rangle := \int_H \Phi(u)\ d\mu(u) = \mathrm{Lim}_{T\to\infty}\frac{1}{T}\int_0^T \Phi(u(t))\ dt, \tag{2.60}$$

where Lim is a Hahn-Banach extension of the classical limit. The measure μ is called a *time-average measure* of u. Note that neither Lim nor μ are unique. The use of Lim surmounts the technical difficulty that the limit in the usual sense may not exist. If u is weak solution to the 2D NSE, then by regularity of such solutions, one can work in the strong topology on H. Moreover, by uniqueness, one can show that μ is in fact *invariant* with respect to the corresponding semigroup, i.e. $\mu(E) = \mu(S(t)^{-1}E)$ for all $t \geq 0$, for all measurable sets $E \subset H$. Thus, a time-average measure is also a so-called *stationary statistical solution* of the NSE. In fact, the support of any time-average measure in 2D is contained in the global attractor, $\mathcal{A}$. Whereas, in 3D, the support of a time-average measure is contained the weak global attractor, $\mathcal{A}_w$, which is defined by

$$\mathcal{A}_w := \{u_0 \in H : \exists u = u(t) \text{ weak solution of NSE}, t\in\mathbb{R},\ \text{uniformly bounded in } H, u(0) = u_0\}.$$

For a more detailed background see [42].

We now specialize to the cases of 3D and 2D turbulence, and interpret the main theorems in those settings.

3D TURBULENCE

The mean energy dissipation rate per unit mass is defined as

$$\varepsilon := \nu\kappa_0^3\langle\|\nabla u\|_{L^2}^2\rangle \ . \tag{2.61}$$

In 3D, Kolmogorov argued that because one can ignore nonlinear effects in the dissipation range, the length scale indicating where dissipation is the dominant effect should depend solely on ε and ν. By a simple dimensional argument, one then arrives at

$$\lambda_\varepsilon = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4} . \tag{2.62}$$

In other words, according to Kolmogorov, for turbulent flows in 3D, $\lambda_d \sim \lambda_\varepsilon$ with λ_ε given in (2.62). We will now describe the best known rigorous result in this direction.

In [31], the radius of analyticity was estimated in terms of ε_{sup} as

$$\lambda_a \gtrsim \frac{(\nu\kappa_0)^3}{\varepsilon_{\text{sup}}}. \tag{2.63}$$

where

$$\varepsilon_{\text{sup}} := \nu\kappa_0^3 \sup_{0\le t\le T^*/2} \|\nabla u(t)\|_{L^2}^2 \tag{2.64}$$

represents the largest instantaneous energy dissipation rate (per unit mass) up to time $T^*/2$, and T^* is the maximal time of existence of a regular solution. A heuristic argument is given to support $\varepsilon_{\text{sup}} \sim \varepsilon$ as in [31]. With this identification, (2.63) becomes

$$\lambda_d \gtrsim \kappa_0^{-1}(\kappa_0 \tilde{\lambda}_\varepsilon)^4 \,, \quad \text{where} \quad \tilde{\lambda}_\varepsilon = \left(\frac{\nu^3}{\varepsilon_{\text{sup}}}\right)^{1/4} \tag{2.65}$$

It is not presently known if ε_{sup} remains finite beyond T^*. Hence, it is not possible to obtain an estimate of the smallest length scale for an arbitrary weak solution. In fact, it is not possible to extend these estimates on the weak attractor either since it is not known whether or not a trajectory, i.e. a weak solution defined for all $t \in \mathbb{R}$, is regular. However, it is well-accepted that statements regarding length scales in turbulence actually concern "averages" and not specific trajectories (cf. [39, 41, 43, 1], or [42, 44] for introductory approaches). Indeed, this is the thrust of our current discussion.

In addition to the dissipation range and wave number, another basic tenet in the Kolmogorov theory of turbulence is the so-called power law for the energy spectrum. More specifically, let $\bar{\kappa}$ denote the wave number in which energy is injected into the flow, i.e., $f = P_{\bar{\kappa}} f$. Denote the Kolmogorov wave-number $\kappa_\varepsilon := 1/\lambda_\varepsilon$. Then the range of wave-numbers $[\bar{\kappa}, \kappa_\varepsilon]$ is known as the inertial range in which the effect of viscosity is negligible. The nonlinear (inertial) term simply transfers the energy injected into the flow through the inertial range at a rate of ε. Moreover, defining the quantity

$$\mathrm{e}_{\kappa_1,\kappa_2} := \kappa_0^3 \langle \|(P_{\kappa_2} - P_{\kappa_1})u\|_{L^2}^2 \rangle,$$

the well-celebrated Kolmogorov's power law asserts that a turbulent flow must satisfy the relation

$$\mathrm{e}_{\kappa,2\kappa} \sim \varepsilon^{2/3}/\kappa^{2/3}, \text{ for } \kappa \in [\bar{\kappa}, \kappa_\varepsilon]. \tag{2.66}$$

Additionally, it is also known that if the Grashof number, G, is sufficiently small, where G is a scalar quantity defined by

$$G := \frac{\kappa_0^{n/2}}{\omega_\alpha^2 \kappa_0^{-1}} \sup_{0 \le t \le T_f} \|f(t)\|_{L^2} \tag{2.67}$$

then the flow is not turbulent and the attractor in this case consists of only one point. In view of this discussion, we *define* a flow to be turbulent if the Kolmogorov power law holds and the Grashof number is sufficiently large, i.e.

$$G \gtrsim \left(\frac{\bar{\kappa}}{\kappa_0}\right)^{3/2}, \tag{2.68}$$

for any dimension $n \geq 1$. One can show that when f is time-independent and has only finitely many modes, i.e. $f = P_{\bar{\kappa}} f$, where

$$P_{\bar{\kappa}} f := \sum_{|k| \le \bar{\kappa}/\kappa_0} \hat{f}(k) e^{i\kappa_0 k \cdot x}, \tag{2.69}$$

then M_f is comparable to G up to a constant depending on only $\kappa_0, \bar{\kappa}$, a fixed parameter τ, and λ_f, where λ_f satisfies

$$\sup_{|y| \le \lambda_f} \|f(\,\cdot + iy)\|_{L^2} < \infty; \tag{2.70}$$

see Proposition 25 in Appendix A.

Now, it is shown in [29] that for such a flow one necessarily has the bounds

$$\frac{\nu^2}{\kappa_0}\left(\frac{\kappa_0}{\bar{\kappa}}\right)^{5/2} G \lesssim \langle \|u\|_{L^2}^2 \rangle \lesssim \frac{\nu^2}{\kappa_0}\left(\frac{\kappa_0}{\bar{\kappa}}\right) G, \tag{2.71}$$

$$\nu^2 \kappa_0 \left(\frac{\kappa_0}{\bar{\kappa}}\right)^{11/4} G^{3/2} \lesssim \langle \|A^{1/2} u\|_{L^2}^2 \rangle \lesssim \nu^2 \kappa_0 \left(\frac{\kappa_0}{\bar{\kappa}}\right)^{1/2} G^{3/2}. \tag{2.72}$$

The following is the main result of this section which recovers the estimate in [31] for 3D turbulent flows.

Theorem 8. Let μ be a time-average measure for a 3D turbulent flow. Then

$$\lambda_d(u) \gtrsim \kappa_0^{-1}(\kappa_0\lambda_\varepsilon)^4$$

holds with probability $1-p$ on the weak attractor, $\mathcal{A}_w$, with respect to μ.

Proof. By definition of $\varepsilon, \lambda_\varepsilon$ and the relation (2.72), we have

$$\kappa_0\lambda_\varepsilon \sim \kappa_0\left(\frac{\nu^3}{\varepsilon}\right)^{1/4} \sim \kappa_0\left(\frac{\nu^3}{\nu\kappa_0^3\langle\|A^{1/2}u\|_{L^2}^2\rangle}\right)^{1/4} \sim \kappa_0\left(\frac{1}{\kappa_0^4 G^{3/2}}\right)^{1/4} \sim G^{-3/8}.$$

In other words, $(\kappa_0\lambda_\varepsilon)^{8/3} \sim G^{-1}$. Since (2.72) and Chebyshev's inequality imply that

$$\mu\left\{u \in \mathcal{A}_w : \nu^{-2}\kappa_0^{-1}\|A^{1/2}u\|_{L^2}^2 \gtrsim p^{-1}G^{3/2}\right\} \leq p,$$

it follows that

$$\mu\left\{u \in \mathcal{A}_w : \nu^{-2}\kappa_0^{-1}\|A^{1/2}u\|_{L^2}^2 \lesssim p^{-1}G^{3/2}\right\} \geq 1-p. \tag{2.73}$$

Hence, Theorem 6 (with $\alpha = 2, r = 1, n = 3, p = 2, \sigma = 1, q' = 12$) implies that the maximal radius of spatial analyticity of trajectories outside this set must satisfy

$$\lambda_d(u) \gtrsim_p \kappa_0^{-1}G^{-3/2} \sim \kappa_0^{-1}(\kappa_0\lambda_\varepsilon)^{-4}, \tag{2.74}$$

holds with probabiilty $1-p$ with respect to μ, as desired. □

Remark 9. We note that there are other ways to identify a small length scale in the flow. Another such way is through the dimension of the attractor, $d_{\mathcal{A}}$, which is related to the number of degrees of freedom in the sense of Landau (see [63]). In this direction, Gibbon and Titi show in [47] that

$$\lambda_d \gtrsim \bar{\lambda}_{\varepsilon}^{1.6}, \tag{2.75}$$

with $\bar{\epsilon}$ defined as in (2.64). In contrast, the estimate of the dissipation length scale in Theorem 8 is associated with the *exponential decay* of the Fourier spectrum, and again, our estimate is in terms of the actual Kolmogorov length scale λ_ε, rather than $\bar{\lambda}_\varepsilon$.

In [7], so-called ladder estimates are used to identify a small length scales in 2D and 3D. However, in 3D their estimates involve the quantity $\|\nabla u\|_{L^\infty}$, as in the work of Henshaw, Kreiss, and Reyna in [51].

2D TURBULENCE

In the Kraichnan theory of 2D turbulence enstrophy $\|A^{1/2}u\|_{L^2}^2$ is also dissipated, and it does so at a mean rate per unit mass given by

$$\eta = \nu\kappa_0^2\langle\|Au\|_{L^2}^2\rangle \ .$$

Two key wave numbers are

$$\kappa_\eta := \left(\frac{\eta}{\nu^2}\right)^{1/6} \sim \left(\frac{\langle\|Au\|_{L^2}^2\rangle}{L^2\nu^2}\right)^{1/6}, \quad \kappa_\sigma := \left(\frac{\langle\|Au\|_{L^2}^2\rangle}{\langle\|A^{1/2}u\|_{L^2}^2\rangle}\right)^{1/2},$$

where A is the Stokes operator.

It is shown in [28], that if the well-recognized power law

$$e_{\kappa,2\kappa} = \langle\|P_{2\kappa}Q_\kappa u\|_{L^2}^2\rangle \sim \frac{\eta^{2/3}}{\kappa^2}, \tag{2.76}$$

holds over the *inertial range* $[\underline{\kappa}_i, \bar{\kappa}_i]$ and if

$$\underline{\kappa}_i \leq 4\kappa_\eta, \quad \langle \|A^{1/2}P_{\underline{\kappa}_i}u\|_{L^2}^2\rangle \lesssim \langle \|A^{1/2}Q_{\underline{\kappa}_i}u\|_{L^2}\rangle, \quad G \gtrsim (\bar{\kappa}/\kappa_0)^2, \tag{2.77}$$

then

$$\nu^2\kappa_0^2\left(\frac{\bar{\kappa}}{\kappa_0}\right)^{-1} G \lesssim \langle \|A^{1/2}u\|_{L^2}^2\rangle \lesssim \nu^2\kappa_0^2\left(\frac{\bar{\kappa}}{\kappa_0}\right) G(\ln G)^{3/2} \tag{2.78}$$

$$\nu^2\kappa_0^4\left(\frac{\bar{\kappa}}{\kappa_0}\right)^{-3/2} \frac{G^{3/2}}{(\ln G)^{3/2}} \lesssim \langle \|Au\|_{L^2}^2\rangle \lesssim \nu^2\kappa_0^4\left(\frac{\bar{\kappa}}{\kappa_0}\right)^{3/2} G^{3/2}(\ln G)^{3/4} . \tag{2.79}$$

This is to say that *on average* $\|A^{1/2}u\|_{L^2}$ is of order $\nu\kappa_0 G^{1/2}$ on the global attractor. As in the 3D case, we can make this precise in terms of probabilities.

First, observe that by the time-averaged Brézis-Gallouët inequality (see Proposition 11)

$$(\nu\kappa_0)^2\langle \|u_0\|_{\mathcal{W}}^2\rangle \lesssim \langle \|A^{1/2}u_0\|_{L^2}^2\rangle \left(1 + \ln\left({\kappa_\sigma}^2/\kappa_0^2\right)\right) .$$

Hence, (2.78) and (2.79) imply that

$$\langle \|u_0\|_{\mathcal{W}}^2\rangle \lesssim \mathcal{L}G,$$

where

$$\mathcal{L} := (\bar{\kappa}/\kappa_0)(\ln G)^{3/2}[1 + \ln(\kappa_\sigma^2/\kappa_0^2)],$$

Chebyshev's inequality then implies that

$$\mu\left\{u \in \mathcal{A} : \|u\|_{\mathcal{W}}^2 \gtrsim p^{-1}\mathcal{L}G\right\} \leq p, \tag{2.80}$$

for any $0 < p < 1$, provided that both (2.76) and (2.77) hold.

Therefore, we can conclude by Theorem 6 that

$$\mu \left\{ u \in \mathcal{A} : \lambda_a \gtrsim_p \kappa_0^{-1} G^{-1/2} \right\} \geq 1 - p, \tag{2.81}$$

where the suppressed constant inside depends only on $p, \bar{\kappa}/\kappa_0$, and logarithms of G. Since by (2.79)

$$\lambda_\eta = \left(\frac{\nu^3}{\eta} \right)^{1/6} \leq \frac{1}{\kappa_0} \left(\frac{\kappa_0}{\bar{\kappa}} \right)^{1/4} G^{-1/4} ,$$

we have the following

Theorem 10. Let μ be a time-invariant measure for a 2D turbulent flow. Then

$$\lambda_d(u) \gtrsim_p \kappa_0^{-1} (\kappa_0 \lambda_\eta)^2$$

holds with probability $1 - p$ on $\mathcal{A}$ with respect to μ.

To prove Theorem 10 we invoked a time-averaged version of the Brézis-Gallouët, whose proof we supply now.

Proposition 11. Let $L > 0$ and $\Omega = [0, L]^2$. Let $\mathcal{A}$ be the global attractor of (2.3) with time-independent forcing f satisfying $P_{\bar{\kappa}} f = f$. Then there exists an absolute constant $C > 0$ such that

$$(\nu\kappa_0)^2 \langle \|u\|_{\mathcal{W}}^2 \rangle \leq C \langle \|A^{1/2} u\|_{L^2(\Omega)}^2 \rangle \left[1 + \ln \left(\kappa_0^{-2} \frac{\langle \|Au\|_{L^2(\Omega)}^2 \rangle}{\langle \|A^{1/2} u\|_{L^2(\Omega)}^2 \rangle} \right) \right], \tag{2.82}$$

for all $u \in \mathcal{A}$, where A is the Stokes operator, and $\langle \cdot \rangle$ denotes an ensemble average in the sense of (2.60).

Proof. Let $u_k := |\hat{u}(k)|$ for all $k \in \mathbb{Z}^n$. Fix $\lambda > 0$ to be chosen later. Observe that

$$\sum_{k\in\mathbb{Z}^d} u_k = \underbrace{\sum_{|k|\le\lambda} |k|^{-1}|k|u_k}_{S_1} + \underbrace{\sum_{|k|>\lambda} |k|^{-2}|k|^2 u_k}_{S_2}.$$

Estimate S_1 with Cauchy-Schwarz to get

$$S_1 \le \left(\sum_{|k|\le\lambda} |k|^{-2}\right)^{1/2} \left(\sum_{|k|\le\lambda} |k|^2 u_k^2\right)^{1/2}.$$

Observe that

$$\sum_{|k|\le\lambda} |k|^{-2} \le C\int_1^\lambda r^{-1}\, dr = C\log\lambda.$$

On the other hand, we estimate S_2 as follows

$$S_2 \le \left(\sum_{|k|>\lambda} |k|^{-4}\right)^{1/2} \left(\sum_{|k|>\lambda} |k|^4 u_k^2\right)^{1/2}.$$

Observe that

$$\sum_{|k|>\lambda} |k|^{-4} \le C\int_\lambda^\infty r^{-3}\, dr = \frac{C}{2}\lambda^{-2}.$$

Combining S_1 and S_2, so far we have

$$\|\mathbf{u}\|_{\ell^1} \le C(\log\lambda)\||\cdot|\mathbf{u}\|_{\ell^2} + \frac{C}{2}\lambda^{-2}\||\cdot|^2\mathbf{u}\|_{\ell^2},$$

An elementary calculation gives

$$\|\mathbf{u}\|_{\ell^1}^2 \le 2C^2(\log\lambda)^2\||\cdot|\mathbf{u}\|_{\ell^2} + \frac{C^2}{2}\lambda^{-4}\||\cdot|^2\mathbf{u}\|_{\ell^2}^2.$$

Taking time-averages, monotonicity and linearity of generalized Banach limits imply

$$\langle \|\mathbf{u}\|_{\ell^1}^2 \rangle \leq C(\log \lambda)\langle \| |\cdot| \mathbf{u}\|_{\ell^2}^2 \rangle + \frac{C}{2}\lambda^{-2}\langle \| |\cdot|^2 \mathbf{u}\|_{\ell^2}^2 \rangle, \tag{2.83}$$

Now choose λ such that

$$\lambda^{-2} = \frac{\langle \| |\cdot| \mathbf{u}\|_{\ell^2}^2 \rangle}{\langle \| |\cdot|^2 \mathbf{u}\|_{\ell^2}^2 \rangle}.$$

Observe that $\lambda \geq 1$. Therefore, for some absolute constant $C > 0$,

$$\langle \|\mathbf{u}\|_{\ell^1}^2 \rangle \leq C\langle \|\mathbf{u}\|_{\ell^2}^2 \rangle \left[1 + \ln\left(\frac{\langle \| |\cdot|^2 \mathbf{u}\|_{\ell^2}^2 \rangle}{\langle \| |\cdot| \mathbf{u}\|_{\ell^2}^2 \rangle}\right)\right].$$

Rescaling with physical units and applying Parseval's identity completes the proof. □

2.3.3 COMPARISON TO ENERGY METHOD

In [31], the Gevrey norm approach of [40] was refined to allow room to optimize the analyticity radius. However, the phase space used there was $L^2(\Omega)^3$. We repeat the calculation here but with the Wiener algebra as the phase space for comparison. We use the velocity formulation Navier-Stokes, although the calculation can be done with the vorticity formulation as well.

Suppose $\lambda(t) : [0, \infty) \to [0, \infty)$ such that $\lambda(0) = 0$ and that λ has physical dimensions length. Suppose f is identically zero, so that $M = M_0$ in (2.20). Let u be a Gevrey regular weak solution to (2.1).

Observe that

$$2|e^{\lambda(t)|\kappa_0 k|}\hat{u}(k)|\frac{d}{dt}|e^{\lambda(t)|\kappa_0 k|}\hat{u}(k)| = 2\lambda'(t)|\kappa_0 k|e^{\lambda(t)|\kappa_0 k|}|\hat{u}(k)|^2 + e^{2\lambda(t)|\kappa_0 k|}\frac{d}{dt}|\hat{u}(k)|^2. \tag{2.84}$$

Now observe that

$$\frac{d}{dt}|\hat{u}(k)|^2 = 2\,\mathrm{Re}\left(\overline{\hat{u}(k)}\frac{d}{dt}\hat{u}(k)\right). \tag{2.85}$$

Since

$$\mathrm{Re}\left(\overline{\hat{u}(k)}\frac{d}{dt}\hat{u}(k)\right) = -\nu|\kappa_0 k|^2|\hat{u}(k)|^2 + \mathrm{Re}\left[i\left(\mathcal{P}\sum_{\ell}(\kappa_0 k)\cdot\hat{u}(\ell)\hat{u}(k-\ell)\right)\overline{\hat{u}(k)}\right], \tag{2.86}$$

we can combine (2.84)-(2.86) to obtain

$$\begin{aligned}\frac{d}{dt}|e^{\lambda(t)|\kappa_0 k|}\hat{u}(k)| =& \lambda'(t)e^{\lambda(t)|\kappa_0 k|}|\kappa_0 k||\hat{u}(k)| - \nu e^{\lambda(t)|\kappa_0 k|}|\kappa_0 k|^2|\hat{u}(k)| \\ &+ \mathrm{Re}\left[i\left(\mathcal{P}\sum_{\ell}(\kappa_0 k)\cdot\hat{u}(\ell)\hat{u}(k-\ell)\right)e^{\lambda(t)||\kappa_0 k|}\frac{\overline{\hat{u}(k)}}{|\hat{u}(k)|}\right].\end{aligned}$$

The divergence-free condition, $k\cdot\hat{u}(k)=0$ for all $k\in\mathbb{Z}^n$, (2.42), then summing over k gives

$$\begin{aligned}\frac{d}{dt}\|e^{\lambda(t)A^{1/2}}u\|_{\mathcal{W}} \leq& \underbrace{\lambda'(t)\|e^{\lambda(t)A^{1/2}}A^{1/2}u\|_{\mathcal{W}}}_{I} - \nu\|e^{\lambda(t)A^{1/2}}Au\|_{\mathcal{W}} \\ &+ 2(\nu\kappa_0)^{-1}\underbrace{\left(\sum_k\sum_\ell|\kappa_0(k-\ell)||\hat{u}(\ell)||\hat{u}(k-\ell)|e^{\lambda t\kappa_0|k|}\right)}_{II}.\end{aligned}$$

We estimate I as follows:

$$\begin{aligned}\lambda'(t)\|e^{\lambda(t)A^{1/2}}A^{1/2}u\|_{\mathcal{W}} &\leq \lambda'(t)\|e^{\lambda(t)A^{1/2}}u\|_{\mathcal{W}}^{1/2}\|e^{\lambda(t)A^{1/2}}Au\|_{\mathcal{W}}^{1/2} \\ &\leq \frac{2\lambda'(t)^2}{\nu}\|e^{\lambda(t)A^{1/2}}u\|_{\mathcal{W}} + \frac{\nu}{2}\|e^{\lambda(t)A^{1/2}}Au\|_{\mathcal{W}}.\end{aligned} \tag{2.87}$$

We estimate II as:

$$
\begin{aligned}
2(\nu\kappa_0)^{-1}(II) &\leq 2(\nu\kappa_0)^{-1}\sum_{\ell} e^{\lambda(t)|\kappa_0 k|}|\hat{u}(\ell)|\sum_{k} e^{\lambda(t)|\kappa_0(k-\ell)|}|\kappa_0(k-\ell)||\hat{u}(k-\ell)| \\
&= 2(\nu\kappa_0)\|e^{\lambda(t)A^{1/2}}u\|_{\mathcal{W}}\|e^{\lambda(t)A^{1/2}}A^{1/2}u\|_{\mathcal{W}} \\
&\leq 2(\nu\kappa_0)\|e^{\lambda(t)A^{1/2}}u\|_{\mathcal{W}}^{3/2}\|e^{\lambda(t)A^{1/2}}Au\|_{\mathcal{W}}^{1/2} \\
&\leq 2(\nu\kappa_0^2)\|e^{\lambda(t)A^{1/2}}u\|_{\mathcal{W}}^{3} + \frac{\nu}{2}\|e^{\lambda(t)A^{1/2}}Au\|_{\mathcal{W}}.
\end{aligned}
\tag{2.88}
$$

Combining (2.87) and (2.88) we get

$$
\frac{d}{dt}\|e^{\lambda(t)A^{1/2}}u\|_{\mathcal{W}} \leq \frac{2\lambda'(t)^2}{\nu}\|e^{\lambda(t)A^{1/2}}u\|_{\mathcal{W}} + 2(\nu\kappa_0^2)\|e^{\lambda(t)A^{1/2}}u\|_{\mathcal{W}}^{3}. \tag{2.89}
$$

Observe that

$$
\begin{aligned}
&\frac{d}{dt}\left(e^{-\frac{2}{\nu}\int_0^t \lambda'(s)^2\ ds}\|e^{\lambda(t)A^{1/2}}u\|_{\mathcal{W}}\right) \\
&= -\frac{2}{\nu}\lambda'(t)^2 e^{-\frac{2}{\nu}\int_0^t \lambda'(s)^2\ ds}\|e^{\lambda(t)A^{1/2}}u\|_{\mathcal{W}} + e^{-\frac{2}{\nu}\int_0^t \lambda'(s)^2\ ds}\frac{d}{dt}\|e^{\lambda(t)A^{1/2}}u\|_{\mathcal{W}} \\
&\leq 2(\nu\kappa_0^2)e^{\frac{4}{\nu}\int_0^t \lambda'(s)^2\ ds}\left(e^{-\frac{2}{\nu}\int_0^t \lambda'(s)^2\ ds}\|e^{\lambda(t)A^{1/2}}u\|_{\mathcal{W}}\right)^3.
\end{aligned}
\tag{2.90}
$$

Then

$$
\|e^{\lambda(t)A^{1/2}}u\|_{\mathcal{W}} \leq \frac{\sqrt{E'(t)}\|u_0\|_{\mathcal{W}}}{\sqrt{1-4\nu\kappa_0^2 E(t)\|u_0\|_{\mathcal{W}}^2}}, \tag{2.91}
$$

where

$$
E(t) := \int_0^t e^{\frac{4}{\nu}\int_0^\tau \lambda'(s)^2\ ds}\ d\tau. \tag{2.92}
$$

Observe that (2.91) is valid for all $0 \leq t \leq T^{**} \wedge T^{*}$, where T^{*} is the existence time guaranteed by Theorem 5, such that

$$1 - 4\nu\kappa_0^2 E(t)\|u_0\|_{\mathcal{W}}^2 > 0. \tag{2.93}$$

If $\lambda(t) = \lambda t$, where λ has the physical dimensions of length/time, then

$$T^{**} = \frac{\nu}{4\lambda^2} \log\left(\frac{\lambda^2}{\nu^2\kappa_0^2} \frac{1}{\|u_0\|_{\mathcal{W}}^2} + 1\right). \tag{2.94}$$

It follows that the maximal radius of spatial analyticity at time $T^{**}/2$ satisfies

$$\lambda_a(T^{**}/2) \geq \frac{\nu}{8\lambda} \log\left(\frac{\lambda^2}{\nu^2\kappa_0^2} \frac{1}{\|u_0\|_{\mathcal{W}}^2} + 1\right). \tag{2.95}$$

We may view $\lambda_a(T^*/2)$ as a function of the parameter λ and optimize with respect to this parameter. If we let

$$\lambda_0 := \sqrt{\gamma}(\nu\kappa_0)\|u_0\|_{\mathcal{W}}, \tag{2.96}$$

where $\gamma \in \mathbb{R}$ is the solution of

$$\frac{1}{2\gamma}\log(1+\gamma) - \frac{1}{1+\gamma} = 0, \tag{2.97}$$

then

$$\frac{d}{d\lambda}\lambda_a(T^{**}/2)|_{\lambda=\lambda_0} = 0 \tag{2.98}$$

and

$$\lambda_a(T^{**}/2)|_{\lambda=\lambda_0} \geq C^{**}\kappa_0^{-1}\frac{1}{\|u_0\|_{\mathcal{W}}}, \tag{2.99}$$

where $C^{**} := \frac{\log(1+\gamma)}{8\sqrt{\gamma}}$. This is precisely the estimate that Theorem 5 gives with $\alpha = 2, r = 1, \sigma = \beta = 0$, and f identically 0, except with a different value for C^{**}. While this method can accommodate for forces, f, with finitely many modes, our approach allows forces with infinitely many modes.

We also remark that the above choice of $\lambda(t) = \lambda t$, although simple, may not be the "optimal" choice. While it does agree with our estimates, this method seems to give some freedom in the choice of the $\lambda(t)$. We note, however, that $\lambda(t) = \sqrt{\nu t}$ is *not* allowed from this method, as it would violate (2.93). It is interesting then that choosing $\lambda(t) = \lambda t$ and optimizing with respect to the free parameter λ is in some sense equivalent to our approach where the scaling is naturally determined by $-\Delta$ (see Proposition 15).

2.4 PROOF OF MAIN THEOREMS

Our goal is to satisfy the hypothesis of Theorem 3. In particular, we will estimate Φ and W as given (2.32), i.e.,

$$(Su(\,\cdot\,))(t) := \underbrace{e^{-\nu_\alpha t A^{\alpha/2}}u_0 + \int_0^t e^{-\nu_\alpha (t-s)A^{\alpha/2}}f(s)\;ds}_{\Phi(t)} - \underbrace{\int_0^t e^{-\nu_\alpha (t-s)A^{\alpha/2}}B_r[u(s),u(s)]\;ds}_{W(u,u)(t)}\,.$$

To do so, we will make use of the following elementary facts.

Lemma 12. For $c > 1$ and $x, y \geq 0$

$$(x \wedge y) \leq (x \wedge cy) \leq c(x \wedge y). \tag{2.100}$$

Proposition 13. Let $a, b, c > 0$ and $1 < \alpha \leq 2$. Then

$$\sup_{x\in\mathbb{R}_+} \left\{x^b e^{-cx^a}\right\} = \left(\frac{b}{ea}\right)^{b/a} c^{-b/a}. \tag{2.101}$$

and

$$\sup_{x\in\mathbb{R}_+} \{ax - bx^\alpha\} = \left(\frac{a^\alpha}{b}\right)^{1/(\alpha-1)} \left(\frac{1}{\alpha}\right)^{1/(\alpha-1)} \left(\frac{\alpha-1}{\alpha}\right). \tag{2.102}$$

We will also need the following estimates regarding the heat kernel, $e^{-\nu_\alpha t A^{\alpha/2}}$.

Proposition 14. Let $1 \leq p \leq \infty$, $\alpha, \beta > 0$, and $\lambda, \sigma \in \mathbb{R}$. Then

$$(\nu_\alpha t)^{\beta/\alpha} \|e^{-\nu_\alpha t A^{\alpha/2}} u\|_{\lambda,\sigma+\beta,p} \leq C_{14}(\alpha,\beta)\|u\|_{\lambda,\sigma,p} \tag{2.103}$$

for all $t > 0$, where

$$C_{14}(\beta,\alpha) := \left(\frac{\beta}{e\alpha}\right)^{\beta/\alpha}.$$

Proposition 15. Let $1 \leq p \leq \infty$, $\sigma \in \mathbb{R}$, and $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ be a sublinear function. Suppose $1 < \alpha \leq 2$. Then

$$\|e^{-\nu_\alpha (t-s) A^{\alpha/2}} u\|_{\lambda(t),\sigma,p} \leq C_{15}(s,t,\alpha,\nu_\alpha)\|e^{-(\nu_\alpha/2)(t-s)A^{\alpha/2}} u\|_{\lambda(s),\sigma,p} \tag{2.104}$$

for all $t > 0$, where

$$C_{15}(s,t,\alpha,\nu_\alpha) := \exp\left[\left(\frac{\lambda(t-s)^\alpha}{\nu_\alpha(t-s)}\right)^{1/(\alpha-1)} \left(\frac{1}{\alpha}\right)^{1/(\alpha-1)} \left(\frac{\alpha-1}{\alpha}\right)\right].$$

If $\alpha = 1$ and we moreover assume

$$\lambda(t) \leq \frac{1}{2}\nu_\alpha t,$$

then

$$\|e^{-\nu_\alpha(t-s)A^{1/2}}u\|_{\lambda(t),\sigma,p} \leq \|e^{-(\nu_\alpha/2)(t-s)A^{1/2}}u\|_{\lambda(s),\sigma,p}$$

for all $t > 0$.

Proposition 16 (Biswas-Swanson)**.** Let $n \geq 1$, $\lambda \geq 0$, and $1 < p < \infty$ with p' its Hölder conjugate. Suppose that $n/(2p') < \gamma < n/p'$.

$$||u * v||_{\lambda,2\gamma-n/p',p} \leq C_{16}(n,\gamma,p)\kappa_0^{-n/p'}||u||_{\lambda,\gamma,p}||v||_{\lambda,\gamma,p}. \tag{2.105}$$

Proposition 17. Let $1 < p < \infty$, $\delta \in \mathbb{R}$, and $r, \lambda, \gamma \geq 0$ such that $n/(2p') < \gamma < n/p'$. Then

$$\|e^{-\nu_\alpha t A^{\alpha/2}}B_r[u,v]\|_{\lambda,\delta,p} \leq C_{17}(n,p,r,\alpha,\gamma,\delta)(\omega_\alpha t)^{-\max\{0,(r+\delta-2\gamma+n/p')/\alpha\}}\kappa_0^{(1+\delta-2\gamma)}\|u\|_{\lambda,\gamma,p}\|v\|_{\lambda,\gamma,p} \tag{2.106}$$

for all $t > 0$, where $\omega_\alpha = \nu_\alpha \kappa_0^\alpha$ and

$$C_{17}(n,p,r,\alpha,\gamma,\delta) = C_{16}(n,p,\gamma)\left(\frac{r+\delta-2\gamma+n/p'}{e\alpha}\right)^{\max\{0,(r+\delta-2\gamma+n/p')/\alpha\}}$$

Proof. We estimate as follows:

$$\begin{aligned}
\|e^{-\nu_\alpha t A^{\alpha/2}} B_r[u,v]\|_{\lambda,\delta,p}^p &= \sum_{k\in\mathbb{Z}^n} e^{-\nu_\alpha t|\kappa_0 k|^\alpha p} e^{\lambda|\kappa_0 k|p} |\kappa_0 k|^{\delta p} \, |B_r[u,v](k)|^p \\
&\leq \kappa_0^{p(1-r)} \sum_{k\in\mathbb{Z}^n} e^{-\nu_\alpha t|\kappa_0 k|^\alpha p} |\kappa_0 k|^{(r+\delta-(2\gamma-n/p'))p} e^{\lambda|\kappa_0 k|p} |\kappa_0 k|^{(2\gamma-n/p')p} (|u| * |v|)(k)^p \\
&\leq \kappa_0^{(1+\delta-(2\gamma-n/p'))p} \|x^{r+\delta-(2\gamma-n/p')} e^{-\nu_\alpha t \kappa_0^\alpha x^\alpha}\|_{L^\infty(\mathbb{R}_+)}^p \||u| * |v|\|_{\lambda,2\gamma-n/p',p}^p \\
&\leq C_{16}^p (\nu_\alpha \kappa_0^\alpha t)^{-p\max\{0,(r+\delta-2\gamma+n/p')\}/\alpha} \kappa_0^{(1+\delta-2\gamma)p} \|u\|_{\lambda,\gamma,p}^p \|v\|_{\lambda,\gamma,p}^p,
\end{aligned}$$

where we have applied Proposition 16 to obtain the last inequality. Raising both sides to the power $1/p$ completes the proof. □

Proposition 18 (Biswas-Swanson)**.** Let $\lambda, \gamma \geq 0$. Then

$$||u * v||_{\lambda,\gamma,1} \leq 2^\gamma \kappa_0^{-\gamma} ||u||_{\lambda,\gamma,1} ||v||_{\lambda,\gamma,1}. \tag{2.107}$$

Proposition 19. Let $\lambda, \gamma \geq 0$. Then for any $\delta \in \mathbb{R}$ and $r \geq 0$

$$\|e^{-\nu_\alpha t A^{\alpha/2}} B_r[u,v]\|_{\lambda,\delta,1} \leq C_{19}(r,\gamma,\delta) \kappa_0^{1+\delta-2\gamma} (\omega_\alpha t)^{-\max\{0,(r+\delta-\gamma)/\alpha\}} \|u\|_{\lambda,\gamma,1} \|v\|_{\lambda,\gamma,1}, \tag{2.108}$$

where

$$C_{19}(r,\alpha,\gamma,\delta) = 2^\gamma \left(\frac{r+\delta-\gamma}{e\alpha}\right)^{\max\{0,(r+\delta-\gamma)/\alpha\}}$$

Proof. Let $\alpha = (1/2)(1+\delta-\gamma)$. We estimate as follows

$$\begin{aligned}
\|e^{-\nu t A} B_r[u,v]\|_{\lambda,\delta} &\leq \sum_{k\in\mathbb{Z}^n} e^{-\nu_\alpha t|\kappa_0 k|^\alpha} e^{\lambda|\kappa_0 k|} |\kappa_0 k|^\delta |B_r[u,v](k)| \\
&\leq \kappa_0^{1-r} \sum e^{-\nu_\alpha t|\kappa_0 k|^\alpha} e^{\lambda|\kappa_0 k|} |\kappa_0 k|^{r+\delta} (|u| * |v|(k)) \\
&\leq \kappa_0^{1+\delta-\gamma} \sum_{k\in\mathbb{Z}^n} e^{-\nu_\alpha t\kappa_0^\alpha |k|^\alpha} |k|^{r+\delta-\gamma} e^{\lambda|\kappa_0 k|} |\kappa_0 k|^\gamma (|u| * |v|(k)) \\
&\leq \kappa_0^{1+\delta-\gamma} \|e^{-\nu_\alpha t\kappa_0^\alpha x^\alpha} x^{r+\delta-\gamma}\|_{\ell^\infty} \||u| * |v|\|_{\lambda,\gamma,1} \\
&\leq C_{19}(r,\alpha,\delta,\gamma)\kappa_0^{1+\delta-2\gamma} (\nu_\alpha \kappa_0^\alpha t)^{-\max\{0,(r+\delta-\gamma)/\alpha\}} \|u\|_{\lambda,\gamma,1} \|v\|_{\lambda,\gamma,1}
\end{aligned}$$

where we have applied Proposition 18 to obtain the last inequality. □

2.4.1 ESTIMATING Φ

Now let us estimate the term

$$\Phi(t) := e^{-\nu_\alpha t A^{\alpha/2}} u_0 + \int_0^t e^{-\nu_\alpha (t-s) A^{\alpha/2}} f(s)\, ds \tag{2.109}$$

for $0 \leq t \leq T$. Recall that ultimately we want $\Phi \in Z_T$ (see (2.24) and (2.32)).

Lemma 20. Let $1 \leq p \leq \infty$ and $1 < q < \infty$. Let $\sigma \in \mathbb{R}$ and $\lambda(t) = \sqrt[\alpha]{\nu_\alpha t}$. Let M and M_f be given as in (2.20). Then for $0 \leq \beta < \alpha/q'$ and a fixed $T \leq T_f$ finite:

1. $\|\Phi\|_X \leq C_{20}^{(i)}(q,\alpha)M$, for $0 \leq t \leq T$ where

$$C_{20}^{(i)}(q,\alpha) = (2/q')^{1/q'} C_{15}(\alpha)$$

2. $\|\Phi\|_Y \leq C_{20}^{(ii)}(p,q,\alpha,\beta)M$, for $0 < t \leq T$ where

$$C_{20}^{(ii)}(p,q,\alpha,\beta) = (2q')^{\beta/\alpha + 1/q'} C_{14}(\alpha,\beta) C_{15}(p,\alpha) C_{24}((\beta q')/\alpha, 0)^{1/q'}$$

Proof. Fix $T \le T_f$ and let $0 \le t \le T$. Observe that

$$\|\Phi(t)\|_{\lambda(t),\sigma,p} \le \underbrace{\|e^{-\nu_\alpha t A^{\alpha/2}} u_0\|_{\lambda(t),\sigma,p}}_{I} + \underbrace{\int_0^t \|e^{-\nu_\alpha (t-s) A^{\alpha/2}} f(s)\|_{\lambda(t),\sigma,p}\, ds}_{II}.$$

We estimate I by applying Proposition 15 with $s = 0$ and using the fact that $e^{-\nu_\alpha t A^{\alpha/2}}$ is a contractive semigroup for $t > 0$ so that

$$\|e^{-\nu_\alpha t A^{\alpha/2}} u_0\|_{\lambda(t),\sigma,p} \le C_{15} \|e^{-(\nu_\alpha/2) t A^{\alpha/2}} u_0\|_\sigma \le C_{15}\|u_0\|_\sigma. \tag{2.110}$$

Now we estimate II. Observe that since f has mean zero, by contractivity and Proposition 15

$$\|e^{-\nu_\alpha (t-s) A^{\alpha/2}} f(s)\|_{\lambda(t),\sigma,p} \le C_{15} \|e^{-(\nu_\alpha/2)(t-s) A^{\alpha/2}} f(s)\|_{\lambda(s)\sigma,p} \le C_{15} e^{-(\omega_\alpha/2)(t-s)} \|f(s)\|_{\lambda(s)\sigma,p}, \tag{2.111}$$

where $\omega_\alpha = \nu_\alpha \kappa_0^\alpha$. Suppose $1 < q < \infty$. Integrating both sides of (2.111) and applying the Hölder inequality gives

$$\int_0^t \|e^{-\nu_\alpha (t-s) A^{\alpha/2}} f(s)\|_{\lambda(t),\sigma,p}\, ds \le (2/q')^{1/q'} C_{15} \omega_\alpha^{-1} \left(\omega_\alpha \int_0^{T_f} \|f(s)\|^q_{\lambda(s),\sigma,p}\, ds \right)^{1/q} \tag{2.112}$$

where q, q' are Hölder conjugates. Adding (2.110), (2.112), normalizing physical dimensions, then taking the supremum over $0 \le t \le T$ proves (i). For $q = \infty$, make an L^1-L^∞ Hölder estimate in (2.111) instead.

To prove (ii), instead let $0 < t \le T$. Observe that

$$\|\Phi(t)\|_{\lambda(t),\sigma+\beta,p} \le \underbrace{\|e^{-\nu_\alpha t A^{\alpha/2}} u_0\|_{\lambda(t),\sigma+\beta,p}}_{I'} + \underbrace{\int_0^t \|e^{-\nu_\alpha (t-s) A^{\alpha/2}} f(s)\|_{\lambda(t),\sigma+\beta,p}\, ds}_{II'}. \tag{2.113}$$

We estimate I' as

$$
\begin{aligned}
\|e^{-\nu_\alpha t A^{\alpha/2}} u_0\|_{\lambda(t),\sigma+\beta,p} &\le C_{15}\|e^{-(\nu_\alpha/2)tA^{\alpha/2}} u_0\|_{\sigma+\beta,p} \\
&\le C_{15}C_{14}(\nu_\alpha t/2)^{-\beta/\alpha}\|u_0\|_{\sigma,p} \\
&\le C_{15}C_{14}(\nu_\alpha/2)^{-\beta/\alpha}(t\wedge\omega_\alpha^{-1})^{-\beta/\alpha}\|u_0\|_{\sigma,p}. \qquad (2.114)
\end{aligned}
$$

Similarly, assuming $1 < q < \infty$, we can estimate II' as

$$
\|e^{-\nu_\alpha (t-s) A^{\alpha/2}} f(s)\|_{\lambda(t),\sigma+\beta,p} \le C_{15}C_{14}e^{-(\omega_\alpha/(2q'))(t-s)}(\nu_\alpha(t-s)/(2q'))^{-\beta/\alpha}\|f(s)\|_{\lambda(s),\sigma,p}. \qquad (2.115)
$$

Now integrate both sides of (2.115), apply the Hölder inequality, then Proposition 24 to obtain

$$
\begin{aligned}
II' \le & C_{15}C_{14}\int_0^t \frac{e^{-(\omega_\alpha/(2q'))(t-s)}}{(\nu_\alpha(t-s)/q')^{\beta/\alpha}}\|f(s)\|_{\lambda(s),\sigma,p}\, ds \qquad (2.116)\\
\le & C_{15}C_{14}C_{24}^{1/q'}\cdot(\nu_\alpha/(2q'))^{-\beta/\alpha}(t\wedge(\omega_\alpha/(2q'))^{-1})^{1/q'-\beta/\alpha}\omega_\alpha^{-1/q}\frac{\kappa_0^\sigma}{\omega_\alpha^{-2}\kappa_0}M_f, \qquad (2.117)
\end{aligned}
$$

where

$$
C_{24}(c,d) = \mathcal{B}(1-c,1-d) = \int_0^1 t^{-c}(1-t)^{-d}\, dt. \qquad (2.118)
$$

An elementary calculation shows that $\mathcal{B}(1-c,1) = \frac{1}{1-c}$, which in particular implies that

$$
C_{24}((\beta q'/\alpha),0) > 1. \qquad (2.119)
$$

Therefore, by adding (2.114) and (2.117) , then applying (2.119) and (2.100) we obtain

$$\nu_\alpha^{\beta/2}\frac{\kappa_0^{-\sigma}}{\omega_\alpha\kappa_0^{-1}}(t\wedge\omega_\alpha^{-1})^{\beta/\alpha}\|\Phi(t)\|_{\lambda(t),\sigma+\beta,p}$$
$$\leq (2q')^{\beta/\alpha+1/q'}C_{15}C_{14}C_{24}^{1/q'}\left(\frac{\kappa_0^{-\sigma}}{\omega_\alpha\kappa_0^{-1}}\|u_0\|_{\sigma,p}+(t\wedge\omega_\alpha^{-1})^{1/q'}\omega_\alpha^{1/q'}M_f\right). \quad (2.120)$$

Using the fact that $(t\wedge\omega_\alpha^{-1})\leq\omega_\alpha^{-1}$, then taking the supremum over $0<t\leq T$ completes the proof of (ii) for $1<q<\infty$.

If $q=\infty$, then instead make an L^1-L^∞ Hölder estimate in (2.116), so that (2.117) becomes

$$\int_0^t\|e^{-\nu_\alpha(t-s)A^{\alpha/2}}f(s)\|_{\lambda(t),\sigma+\beta,p}\,ds\leq C_{15}C_{14}C_{24}(\nu_\alpha/2)^{-\beta/\alpha}(t\wedge(\omega_\alpha/2)^{-1})^{1-\beta/\alpha}\frac{\kappa_0^{\sigma}}{\omega_\alpha^{-2}\kappa_0}M_f,$$

Then apply (2.100) again.

Finally, we prove (iii). By Proposition 15, for $0<t<\omega_\alpha^{-1}$ we have

$$(\nu_\alpha t)^{\beta/\alpha}\|\Phi(t)\|_{\lambda(t),\sigma+\beta,p}$$
$$\lesssim(\nu_\alpha t)^{\beta/\alpha}\|e^{-(\nu_\alpha/2)tA^{\alpha/2}}u_0\|_{\sigma+\beta,p}+(\nu_\alpha t)^{\beta/\alpha}\left(\int_0^t\|e^{-(\nu_\alpha/2)(t-s)A^{\alpha/2}}f(s)\|_{\lambda(s),\sigma+\beta,p}\,ds\right).$$

Now consider the projection P_κ onto modes $|k|\leq\kappa/\kappa_0$ with $Q_\kappa=I-P_\kappa$. Observe that

$$\|e^{-(\nu_\alpha/2)tA^{\alpha/2}}u_0\|_{\sigma+\beta,p}\leq\|e^{-(\nu_\alpha/2)tA^{\alpha/2}}Q_\kappa u_0\|_{\sigma+\beta}+\|e^{-(\nu_\alpha/2)tA^{\alpha/2}}P_\kappa u_0\|_{\sigma+\beta,p}$$
$$\lesssim C_{14}(\nu_\alpha t)^{-\beta/\alpha}\|Q_\kappa u_0\|_{\sigma,p}+\|P_\kappa u_0\|_{\sigma+\beta,p}.$$

Similarly

$$(\nu_\alpha t)^{\beta/\alpha}\|e^{-(\nu_\alpha/2)(t-s)A^{\alpha/2}}f(s)\|_{\lambda(t),\sigma+\beta,p}\lesssim C_{14}\|Q_\kappa f(s)\|_{\lambda(s),\sigma,p}+(\nu_\alpha t)^{\beta/2}\|P_\kappa f(s)\|_{\lambda(s),\sigma+\beta,p}.$$

Since κ is arbitrary, sending $t \to 0^+$ completes the proof.

□

Corollary 21. Under the same hypotheses as Lemma 20, suppose moreover that

$$M_0 \leq C_0 (T\omega_\alpha)^{1/q'} M_f \tag{2.121}$$

for some $C > 0$, where $T \leq T_f$. Then

(i) $\|\Phi\|_X \leq C_{20}^{(i)}(q, \alpha) C_0 (T\omega_\alpha)^{1/q'} M_f$,

(ii) $\|\Phi\|_Y \leq C_{20}^{(ii)}(p, q, \alpha, \beta) C_0 (T\omega_\alpha)^{1/q'} M_f$.

Proof. First, recall (2.111) from the proof of Lemma 20 (i)

$$\|e^{-\nu_\alpha (t-s) A^{\alpha/2}} f(s)\|_{\lambda(t),\sigma,p} \leq C_{15} e^{-(\nu_\alpha/2)(t-s)\kappa_0^\alpha} \|f(s)\|_{\lambda(s),\sigma,p}. \tag{2.122}$$

Since $s \leq t$, we have $e^{-(\nu_\alpha/4)(t-s)\kappa_0^\alpha} \leq 1$. Thus, by integrating (2.122) and applying Hölder's inequality

$$\frac{\kappa_0^{-\sigma}}{\omega_\alpha \kappa_0^{-1}} \int_0^t \|e^{-\nu_\alpha (t-s) A^{\alpha/2}} f(s)\|_{\lambda(t),\sigma,p} \, ds \leq (2/q')^{1/q'} C_{15} (T\omega_\alpha)^{1/q'} M_f. \tag{2.123}$$

After adding (2.110) and applying (2.121), normalizing finishes the proof of (i).

On the other hand, recall (2.120) in the proof of Lemma 20 (ii), which we rewrite as

$$\nu_\alpha^{\beta/\alpha} \frac{\kappa_0^{-\sigma}}{\omega_\alpha \kappa_0^{-1}} (t \wedge \omega_\alpha^{-1})^{\beta/2} \|\Phi(t)\|_{\lambda(t),\sigma+\beta,p} \leq C_{20}^{(ii)} \left(M_0 + (T \wedge \omega_\alpha^{-1})^{1/q'} \omega_\alpha^{1/q'} M_f \right), \tag{2.124}$$

for all $0 < t \leq T$. Therefore, (2.121) and the fact that $(T \wedge \omega_\alpha^{-1}) \leq T$ proves (ii). □

2.4.2 ESTIMATING $W(u,u)(t)$

Lemma 22. Let $1 < p < \infty$ and $1 \le r < \alpha \le 2$. Suppose that $\sigma, \beta \in \mathbb{R}$ satisfy

$$\frac{n}{p'} - (\alpha - r) \le \sigma < \frac{n}{p'} \tag{2.125}$$

$$\max\{0, \frac{n}{2p'} - \sigma\} < \beta < \min\{\frac{\alpha}{2}, \frac{n}{p'} - \sigma\}. \tag{2.126}$$

Then

$$\|W(u,v)\|_Z \le C_{22}(n,p,r,\alpha,\beta,\sigma)\omega_\alpha^{\frac{(\alpha-r)-n/p'+\sigma}{\alpha}}(T \wedge \omega_\alpha^{-1})^{\frac{(\alpha-r)-n/p'+\sigma}{\alpha}}\|u\|_Y\|v\|_Y$$

where

$$C_{22}(n,p,r,\alpha,\beta,\sigma) = C_{15}(\alpha)\max\{C'_{22}(n,p,r,\alpha,\gamma,\sigma,\sigma), C'_{22}(n,p,r,\alpha,\gamma,\gamma,\sigma)\},$$

where C'_{22} is defined by (2.132).

Proof. Let $\gamma := \sigma + \beta$. First, observe that (2.126) implies $n/(2p') < \gamma < n/p'$. On the other hand (2.125) implies for $\delta = \sigma$ or $\delta = \gamma$ that

$$0 < \frac{r + \delta - 2\gamma + n/p'}{\alpha} < 1. \tag{2.127}$$

Indeed

$$r + \frac{n}{p'} - \alpha - \sigma \le \frac{1}{2}\left(r + \frac{n}{p'} - \alpha - \sigma\right) \le 0 < \beta < \min\{\frac{\alpha}{2}, \frac{n}{p'} - \sigma\} \le \frac{1}{2}\min\{\alpha, r + \frac{n}{p'} - \sigma\},$$

from which one can deduce (2.127).

Now let us estimate $\|W(u,v)(t)\|_{\lambda(t),\delta,p}$ for $t > 0$ with $\delta = \sigma$ or $\delta = \sigma + \beta$. We proceed as follows:

$$
\begin{aligned}
\|W(u,v)(t)\|_{\lambda(t),\delta,p} &\le \int_0^t \|e^{-\nu_\alpha(t-s)A^{\alpha/2}} B_r[u(s),v(s)]\|_{\lambda(t),\delta,p}\, ds \\
&\le \int_0^t e^{-(\omega_\alpha/2)(t-s)} \|e^{-(\omega_\alpha/2)(t-s)} B_r[u(s),v(s)]\|_{\lambda(t),\delta,p}\, ds \\
&\le C_{15}(\alpha) \int_0^t e^{-(\omega_\alpha/2)(t-s)} \|e^{-(\omega_\alpha/4)(t-s)} B_r[u(s),v(s)]\|_{\lambda(s),\delta,p}\, ds \\
&\le C_{15} C_{17}(n,p,\alpha,\gamma,\delta) \left(\frac{\omega_\alpha}{4}\right)^{-\frac{r+\delta-2\gamma+n/p'}{\alpha}} \kappa_0^{1+\delta-2\gamma} \nu_\alpha^{-\frac{2\beta}{\alpha}} \left(\frac{\kappa_0^{-\sigma}}{\omega_\alpha \kappa_0^{-1}}\right)^{-2} I(t)\|u\|_Y \|v\|_Y,
\end{aligned}
$$

where $I(t)$ is defined by

$$
I(t) := \int_0^t \frac{e^{-(\omega_\alpha/2)(t-s)}}{(t-s)^{(r+\delta-2\gamma+n/p')/\alpha}(s\wedge \omega_\alpha^{-1})^{2(\gamma-\sigma)/\alpha}}\, ds.
$$

Proposition 24 and (2.100) then implies

$$
I(t) \le C_{24}\left(\frac{r+\delta-2\gamma+n/p'}{\alpha}, \frac{2(\gamma-\sigma)}{\alpha}\right) 2^{\frac{(\alpha-r)-n/p'+\sigma}{\alpha}} (T\wedge \omega_\alpha^{-1})^{\frac{(\alpha-r)-n/p'+\sigma}{\alpha}} (t\wedge (\omega_\alpha/2)^{-1})^{\frac{\sigma-\delta}{\alpha}}.
$$

Hence

$$
\nu_\alpha^{\frac{\delta-\sigma}{\alpha}} \frac{\kappa_0^{-\sigma}}{\omega_\alpha \kappa_0^{-1}} (t\wedge \omega_\alpha^{-1})^{\frac{\delta-\sigma}{\alpha}} \|W(u,v)(t)\|_{\lambda(t),\delta,p} \le C_{15} C_{22}' \omega_\alpha^{\frac{(\alpha-r)-n/p'+\sigma}{\alpha}} (T\wedge \omega_\alpha^{-1})^{\frac{(\alpha-r)-n/p'+\sigma}{\alpha}} \|u\|_Y \|v\|_Y, \tag{2.128}
$$

where we have again applied (2.100) and

$$
C_{22}'(n,p,r,\alpha,\gamma,\delta,\sigma) = C_{17}(n,p,r,\alpha,\delta,\gamma) C_{24}\left(\frac{r+\delta-2\gamma+n/p'}{\alpha}, \frac{2(\gamma-\sigma)}{\alpha}\right) 2^{\frac{(\alpha-r)-n/p'+\sigma}{\alpha}} 4^{\frac{r+\delta-2\gamma+n/p'}{\alpha}}. \tag{2.129}
$$

We may now set $\delta = \sigma$ or γ in (2.131), then take the supremum over $0 \le t \le T$ (since $w(0) = 0$) or $0 < t \le T$, respectively.

□

A similar result holds for the case $p = 1$.

Lemma 23. Let $1 \le r < \alpha \le 2$. Suppose $\sigma, \beta \in \mathbb{R}$ satisfy $\sigma_- \le \beta < \min\{r, \alpha - r, \alpha/2\}$, where $\sigma_- = \max\{0, -\sigma\}$. Then

$$\|W(u,v)\|_Z \le C_{23}(r,\alpha,\gamma,\sigma)\omega_\alpha^{\frac{(\alpha-r)-\beta}{\alpha}}(T \wedge \omega_\alpha^{-1})^{\frac{(\alpha-r)-\beta}{\alpha}}\|u\|_Y\|v\|_Y,$$

where

$$C_{23}(r,\alpha,\gamma,\sigma) = \max\{C'_{23}(r,\alpha,\gamma,\gamma,\sigma), C'_{23}(r,\alpha,\gamma,\sigma,\sigma)\}$$

Proof. Observe that for r, σ, β given as described above we have $\sigma + \beta \ge 0$, $\alpha - r - \beta > 0$, as well as

$$0 \le \frac{r + \delta - \gamma}{\alpha} < 1 \text{ and } 0 \le \frac{2\beta}{\alpha} < 1 \tag{2.130}$$

for $\delta = \sigma$ or $\delta = \sigma + \beta$. Now, as before, we estimate $\|W(u,v)(t)\|_{\lambda(t),\delta,1}$ for $t > 0$ with $\delta = \sigma$ or $\delta = \sigma + \beta$.

Let $\gamma := \sigma + \beta$. Then following the proof of Lemma 22 we get

$$\|W(u,v)(t)\|_{\lambda(t),\delta,1} \le C_{15}C_{19}(r,\alpha,\gamma,\delta)\left(\frac{\omega_\alpha}{4}\right)^{-\frac{r+\delta-\gamma}{\alpha}}\kappa_0^{1+\delta-2\gamma}\nu_\alpha^{-\frac{2\beta}{\alpha}}\left(\frac{\kappa_0^{-\sigma}}{\omega_\alpha\kappa_0^{-1}}\right)^{-2} I(t)\|u\|_Y\|v\|_Y,$$

where

$$I(t) := \int_0^t \frac{e^{-(\omega_\alpha/2)(t-s)}}{(t-s)^{(r+\delta-\gamma)/\alpha}(s \wedge \omega_\alpha^{-1})^{2(\gamma-\sigma)/\alpha}}\, ds.$$

Since (2.130) holds, we may apply Proposition 24, so that (2.100) implies

$$I(t) \leq C_{24}\left(\frac{r+\delta-\gamma}{\alpha}, \frac{2(\gamma-\sigma)}{\alpha}\right) 2^{\frac{(\alpha-r)-\gamma+\sigma}{\alpha}} (T \wedge \omega_\alpha^{-1})^{\frac{(\alpha-r)-\gamma+\sigma}{\alpha}} (t \wedge (\omega_\alpha/2)^{-1})^{\frac{\sigma-\delta}{\alpha}}.$$

Hence

$$\nu_\alpha^{\frac{\delta-\sigma}{\alpha}} \frac{\kappa_0^{-\sigma}}{\omega_\alpha \kappa_0^{-1}} (t \wedge \omega_\alpha^{-1})^{\frac{\delta-\sigma}{\alpha}} \|W(u,v)(t)\|_{\lambda(t),\delta,1} \leq C_{15} C'_{23} \omega_\alpha^{\frac{(\alpha-r)-\gamma+\sigma}{\alpha}} (T \wedge \omega_\alpha^{-1})^{\frac{(\alpha-r)-\gamma+\sigma}{\alpha}} \|u\|_Y \|v\|_Y, \tag{2.131}$$

where we have again applied (2.100) and where

$$C'_{23}(r,\alpha,\gamma,\delta,\sigma) = C_{19}(r,\alpha,\delta,\gamma) C_{24}\left(\frac{r+\delta-\gamma}{\alpha}, \frac{2(\gamma-\sigma)}{\alpha}\right) 2^{\frac{(\alpha-r)-\gamma+\sigma}{\alpha}} 4^{\frac{r+\delta-\gamma}{\alpha}}. \tag{2.132}$$

We may now set $\delta = \sigma$ or γ in (2.131), then take the supremum over $0 \leq t \leq T$ (since $w(0) = 0$) or $0 < t \leq T$, respectively. The fact that $\gamma = \sigma + \beta$ then completes the proof. □

2.4.3 PROOFS OF THEOREMS 4-7

First we prove Theorem 4.

Proof of Theorem 4. Let $\sigma, \beta \in \mathbb{R}$ be given such that they satisfy

$$\frac{n}{p'} - (\alpha - r) < \sigma < \frac{n}{p'},$$
$$\max\{0, \frac{n}{2p'} - \sigma\} < \beta < \min\{\frac{\alpha}{2}, \frac{\alpha}{q'}, \frac{n}{p'} - \sigma\}.$$

Then let X_T, Y_T, Z_T be given by (2.22), (2.23), (2.24).

First, we will apply Theorem 3 to show that such a mild solutions exists. Then we will show that this solution is also a weak solution. To this end, observe that Lemma 20 ensures that $\Phi \in Z$ and $\|\Phi\|_Y \leq C_{20}^{(ii)} M$, so let

$$E_T := \{u \in Z : \|u - \Phi\|_Z \leq C_{20}^{(ii)} M\}. \tag{2.133}$$

On the other hand, Lemma 22 ensures that $W(u,v) : Y \times Y \to Z$ and in fact, that

$$\|W(u,v)\|_Z \leq C_{22} \omega_\alpha^{\frac{\alpha - r - n/p' + \sigma}{\alpha}} (T \wedge \omega_\alpha^{-1})^{\frac{\alpha - r - n/p' + \sigma}{\alpha}} \|u\|_Y \|v\|_Y, \tag{2.134}$$

for all $u, v \in Z$. Now for $u \in E$ and $v \in Z$ observe that

$$\|u\|_Y \leq \|u - \Phi\|_Y + \|\Phi\|_Y \leq 2C_{20}^{(ii)} M. \tag{2.135}$$

Combining (2.134) and (2.135) implies

$$\|W(u,v)\|_Z \leq 2C_{20}^{(ii)} C_{22} (T\omega_\alpha)^{\frac{\alpha - r - n/p' + \sigma}{\alpha}} M \|v\|_Y. \tag{2.136}$$

To apply Theorem 3, we require

$$2C_{20}^{(ii)} C_{22} (T\omega_\alpha)^{\frac{\alpha - r - n/p' + \sigma}{\alpha}} M \leq 1/3.$$

Thus, it suffices to have

$$T\omega_\alpha \leq (C^*)^\alpha M^{-\frac{\alpha}{\alpha - r - n/p' + \sigma}}, \tag{2.137}$$

where $C^* > 0$ is given by

$$C^* := ((1/3)(2 \cdot C_{20}^{(ii)} C_{22})^{-1})^{\frac{1}{\alpha - r - n/p' + \sigma}}. \tag{2.138}$$

Finally, define T^* by

$$T^*\omega_\alpha = (C^*)^\alpha M^{-\frac{\alpha}{\alpha-r-n/p'+\sigma}}. \tag{2.139}$$

Theorem 3 then guarantees that there exists a mild solution $u \in E_{T^*}$ to (2.1). This implies that

$$\frac{\kappa_0^{-\sigma}}{\omega_\alpha \kappa_0^{-1}} \sup_{0\le t\le T^*} \|u(t)\|_{\sqrt[\alpha]{\nu_\alpha t},\sigma,p} < \infty,$$

and hence, that u is Gevrey regular. In particular, the maximal radius of spatial analyticity at time T^* satisfies the lower bound

$$\lambda_a(T^*) \geq \sqrt[\alpha]{\nu_\alpha T^*} = C^*\kappa_0^{-1} M^{\frac{\alpha}{\alpha-r-n/p'+\sigma}} \tag{2.140}$$

since $\omega_\alpha = \nu_\alpha \kappa_0^\alpha$.

Now we show that u is indeed a weak solution. This amounts to proving that $\hat{u}(k,t)$ is differentiable in t a.e. in $[0, T^*]$, for each $k \in \mathbb{Z}^n$. Indeed, since $u \in Z_{T^*}$ is a mild solution to (2.1), we know that

$$\hat{u}(k,t) = e^{-\nu_\alpha t|\kappa_0 k|^{\alpha/2}}\hat{u}_0(k) + \int_0^t e^{-\nu_\alpha(t-s)|\kappa_0 k|^{\alpha/2}}\hat{f}(k,s)\ ds - \kappa_0^{1-r}\int_0^t e^{-\nu_\alpha(t-s)|\kappa_0 k|^{\alpha/2}} B_r[u(s),u(s)](k)\ ds.$$

Firstly, it is clear that $e^{\nu_\alpha(\ \cdot)|\kappa_0 k|^{\alpha/2}}\hat{u}_0(k) \in L^1(0,T^*)$ since $\hat{u}_0(k)$ is constant in t for all $k \in \mathbb{Z}^n$. Now let $\lambda(t) = \sqrt[\alpha]{\nu_\alpha t}$. Since $M_f < \infty$ and $\hat{f}(\mathbf{0}) = \mathbf{0}$, we have

$$\int_0^t \left| e^{-\nu_\alpha(t-s)|\kappa_0 k|^{\alpha/2}}\hat{f}(k,s)\ \right|\ ds \leq t^{1/q'}|\kappa_0 k|^{-\sigma}\int_0^t \|f(s)\|_{\lambda(s),\sigma,p}^q\ ds$$

for each $k \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$. Similarly, for any $D > r - 2\sigma + n/p'$, since $B_r[u(t),u(t)](\mathbf{0}) = \mathbf{0}$ for all t (by (2.2)), Proposition 17 implies

$$\int_0^t \left| e^{-\nu_\alpha (t-s)|\kappa_0 k|^{\alpha/2}} B_r[u(s), u(s)](k) \right| ds \le t|\kappa_0 k|^D \sup_{0\le s\le t} \|e^{-\nu_\alpha (t-s)A^{\alpha/2}} B_r[u(s), v(s)]\|_{\lambda(s),-D,p} < \infty, \tag{2.141}$$

since u is Gevrey regular. Therefore, by the fundamental theorem of calculus, we have

$$\frac{d}{dt}\hat{u}(k,t) = -\nu_\alpha |\kappa_0 k|^{\alpha/2}\hat{u}(k,t) + \hat{f}(k,t) - \kappa_0^{1-r} B_r[u(t), u(t)](k), \tag{2.142}$$

for all $k \in \mathbb{Z}^n$ and a.e. $t \in [0, T^*]$. This completes the proof of Theorem 4. □

The proof of Theorem 5 is similar.

Proof of Theorem 5. Let $1 < q \le \infty$ and q, q' be Hölder conjugates. Suppose $\sigma, \beta \in \mathbb{R}$ satisfy $\sigma_- \le \beta < \min\{r, \alpha - r, \alpha/2, \alpha/q'\}$, where $\sigma_- := \max\{0, -\sigma\}$. Then let X_T, Y_T, Z_T be given by (2.22),(2.23),(2.24).

We proceed as in the previous proof, except apply Lemma 23 in place of Lemma 22. Then (2.134) becomes

$$\|W(u,v)\|_Z \le C_{23}\omega_\alpha^{\frac{(\alpha-r)-\beta}{\alpha}}(T \wedge \omega_\alpha^{-1})^{\frac{(\alpha-r)-\beta}{\alpha}}\|u\|_Y\|v\|_Y \tag{2.143}$$

Combining (2.143) and (2.135) implies

$$\|W(u,v)\|_Z \le 2C_{20}^{(ii)} C_{23}(T\omega_\alpha)^{\frac{\alpha-r-n/p'+\sigma}{\alpha}} M\|v\|_Y. \tag{2.144}$$

To apply Theorem 3, we require

$$2C_{20}^{(ii)} C_{23}(T\omega_\alpha)^{\frac{\alpha-r-\beta}{\alpha}} M \le 1/3.$$

Thus, it suffices to have

$$T\omega_\alpha \leq (C^*)^\alpha M^{-\frac{\alpha}{\alpha-r-\beta}}, \tag{2.145}$$

where $C^* > 0$ is given by

$$C^* := ((1/3)(2\cdot C_{20}^{(ii)} C_{23})^{-1})^{\frac{1}{\alpha-r-\beta}}. \tag{2.146}$$

The proof that u is also a weak solution follows as before, except we choose $D > \gamma - r$ in (2.141) and apply Proposition 19 in place of Proposition 17. □

Now we will prove Theorem 6.

Proof of Theorem 6. Fix $C_* > 0$ and define T^* by

$$T^*\omega_\alpha = (\varepsilon C_*)^{q'} M_f^{-\frac{\alpha}{\alpha-r-n/p'+\sigma+\alpha/q'}},$$

where $\varepsilon > 0$ is chosen so that

$$2C_{20}^{(ii)} C_{22}(C_*)^{\frac{\alpha-r-n/p'+\sigma+\alpha/q'}{\alpha/q'}} \varepsilon^{\frac{\alpha-r-n/p'+\sigma}{\alpha/q'}} < 1/3.$$

Then observe that (2.37) is equivalent to

$$M_0 \leq C_* M_f^{\frac{(\alpha-r)-n/p'+\sigma}{(\alpha-r)-n/p'+\sigma+\alpha/q'}} = ((C_*)^{q'} M_f^{-\frac{\alpha}{(\alpha-r)-n/p'+\sigma+\alpha/q'}})^{1/q'} M_f = \varepsilon^{-1}(T^*\omega_\alpha)^{1/q'} M_f.$$

We proceed as before, except apply Corollary 21 in place of Lemma 20. Indeed, Corollary

21 shows that $\|\Phi\|_Y \leq (T^*\omega_\alpha)^{1/q'} M_f$, so define $E_T \subset Z_T$ by

$$E_T := \{u \in Z_T : \|u - \Phi\|_Z \leq C_{20}^{(ii)} \varepsilon^{-1} (T^*\omega_\alpha)^{1/q'} M_f\}.$$

By Lemma 22 we have

$$\|W(u,v)\|_Z \leq C_{22} \omega_\alpha^{\frac{\alpha - r - n/p' + \sigma}{\alpha}} (T^* \wedge \omega_\alpha^{-1})^{\frac{\alpha - r - n/p' + \sigma}{\alpha}} \|u\|_Y \|v\|_Y.$$

For $u \in E$ and $v \in Z$ we have

$$\begin{aligned}
\|W(u,v)\|_Z &\leq 2C_{20}^{(ii)} C_{22} \varepsilon^{-1} (T^*\omega_\alpha)^{\frac{\alpha - r - n/p' + \sigma + \alpha/q'}{\alpha}} M_f \|v\|_Y \\
&= 2C_{20}^{(ii)} C_{22} \varepsilon^{-1} (\varepsilon C_*)^{\frac{\alpha - r - n/p' + \sigma + \alpha/q'}{\alpha/q'}} \|v\|_Y \\
&= 2C_{20}^{(ii)} C_{22} (C_*)^{\frac{\alpha - r - n/p' + \sigma + \alpha/q'}{\alpha/q'}} \varepsilon^{\frac{\alpha - r - n/p' + \sigma}{\alpha/q'}} \|v\|_Y \\
&\leq (1/3) \|v\|_Y.
\end{aligned}$$

We may now apply Theorem 3 and complete the proof as we did in Theorem 4 with C^* defined by $C^* := (\varepsilon C_*)^{q'/\alpha}$. In particular, if $T^*\omega_\alpha \geq 1$, then $\lambda_a(T^*) \geq \kappa_0^{-1}$, and if $T^*\omega_\alpha < 1$, then $M_f > (C^*)^{\alpha - r - n/p' + \sigma + \alpha/q'}$ and we have

$$\lambda_a(T^*) \geq C^* \kappa_0^{-1} M_f^{-\frac{1}{\alpha - r - n/p' + \sigma + \alpha/q'}}.$$

□

Finally, we prove Theorem 7.

Proof of Theorem 7. The proof follows that of Theorem 6, except that we apply Lemma 23 in place of Lemma 22.

Fix $C_* > 0$ and define T^* by

$$T^*\omega_\alpha = (\varepsilon C_*)^{q'} M_f^{-\frac{\alpha}{(\alpha-r)-\beta+\alpha/q'}},$$

where $\varepsilon > 0$ is chosen so that

$$2C_{20}^{(ii)} C_{23} (C_*)^{\frac{(\alpha-r)-\beta+\alpha/q'}{\alpha/q'}} \varepsilon^{\frac{(\alpha-r)-\beta}{\alpha/q'}} < 1/3.$$

Then observe that (2.39) is equivalent to

$$M_0 \leq C_* M_f^{\frac{(\alpha-r)-n/p'+\sigma}{(\alpha-r)-n/p'+\sigma+\alpha/q'}} = ((C_*)^{q'} M_f^{-\frac{\alpha}{(\alpha-r)-n/p'+\sigma+\alpha/q'}})^{1/q'} M_f = \varepsilon^{-1}(T^*\omega_\alpha)^{1/q'} M_f.$$

Corollary 21 still shows that $\|\Phi\|_Y \leq C_{20}^{(ii)} \varepsilon^{-1}(T^*\omega_\alpha)^{1/q'} M_f$, so define $E_T \subset Z_T$ by

$$E_T := \{u \in Z_T : \|u - \Phi\|_Z \leq C_{20}^{(ii)} \varepsilon^{-1} (T^*\omega_\alpha)^{1/q'} M_f\}.$$

By Lemma 23 we have

$$\|W(u,v)\|_Z \leq C_{23} \omega_\alpha^{\frac{(\alpha-r)-\beta}{\alpha}} (T \wedge \omega_\alpha^{-1})^{\frac{(\alpha-r)-\beta}{\alpha}} \|u\|_Y \|v\|_Y$$

For $u \in E$ and $v \in Z$ we have

$$\begin{aligned}
\|W(u,v)\|_Z &\leq 2C_{20}^{(ii)} C_{23} \varepsilon^{-1} (T^*\omega_\alpha)^{\frac{(\alpha-r)-\beta+\alpha/q'}{\alpha}} M_f \|v\|_Y \\
&= 2C_{20}^{(ii)} C_{23} \varepsilon^{-1} (\varepsilon C_*)^{\frac{(\alpha-r)-\beta+\alpha/q'}{\alpha/q'}} \|v\|_Y \\
&= 2C_{20}^{(ii)} C_{23} (C_*)^{\frac{(\alpha-r)-\beta+\alpha/q'}{\alpha/q'}} \varepsilon^{\frac{(\alpha-r)-\beta}{\alpha/q'}} \|v\|_Y \\
&\leq (1/3)\|v\|_Y.
\end{aligned}$$

We may now apply Theorem 3 and complete the proof as we did in Theorem 4 with C^* defined by $C^* := (\varepsilon C_*)^{q'/\alpha}$. In particular, if $T^*\omega_\alpha \geq 1$, then $\lambda_a(T^*) \geq \kappa_0^{-1}$, and if $T^*\omega_\alpha < 1$, then $M_f > (C^*)^{\alpha - r - \beta + \alpha/q'}$ and we have

$$\lambda_a(T^*) \geq C^* \kappa_0^{-1} M_f^{-\frac{1}{\alpha - r - \beta + \alpha/q'}}.$$

□

2.5 APPENDIX A

Let us first prove the abstract existence theorem that we invoked in order to prove Theorems 4 and 6.

Proof of Theorem 3. Consider the map

$$(Su)(t) = \Phi(t) - W(u,u)(t). \tag{2.147}$$

First we show that $S : E \to E$. Indeed, let $u \in E \subset Z$ and observe that by assumption $W(u, \cdot) : Y \to Z$ is a bounded linear operator with operator norm less than $1/N$ for some $N > 3(1 + \|i\|_{Z\to Y})$. Thus

$$\|Su - \Phi\|_Z \leq \|W(u,u)\|_Z \leq (1/N)\|u\|_Y. \tag{2.148}$$

Since $u \in E$, we have

$$\|u\|_Y \leq \|u - \Phi\|_Y + \|\Phi\|_Y \leq \|i\|_{Z\to Y}\|u - \Phi\|_Z + C \leq C(1 + \|i\|_{Z\to Y}). \tag{2.149}$$

Combining (2.148) and (2.149) gives

$$\|Su - \Phi\|_Z \leq (1/3)C \leq C.$$

Hence $Su \in E$.

Now we prove that Su is a contraction. Indeed, since B is bilinear, we have

$$B[u, u] - B[v, v] = B[u - v, u] + B[v, u - v],$$

which implies

$$Su - Sv = -W(u - v, u) - W(v, u - v).$$

Since $u, v \in E$ implies $u - v \in Y$, we therefore have

$$\|Su - Sv\|_Z \leq (1/N)\|u - v\|_Y + (1/N)\|u - v\|_Y \leq (2/N)\|u - v\|_Z \leq (2/3)\|u - v\|_Z,$$

as desired. □

Proposition 24. Let $b \geq 0$ and $0 \leq c, d < 1$. Then for all $t > 0$

$$\int_0^t \frac{e^{-b(t-s)}}{(t-s)^c (s \wedge b^{-1})^d} \, ds \leq C_{24}(c, d)(t \wedge b^{-1})^{1-c-d}, \tag{2.150}$$

where $C_{24}(c, d) = \max\{\mathcal{B}(1 - c, 1 - d), \Gamma(1 - c)\}$, where Γ is the gamma function and $\mathcal{B}$ is the beta function.

Proof. Firstly, if $b = 0$, then set $(x \wedge b^{-1}) = x$.

Observe that

$$\int_0^t \frac{e^{-b(t-s)}}{(t-s)^c(s\wedge b^{-1})^d}\ ds \le \int_0^t \frac{1}{(t-s)^c s^d}\ ds = t^{-c-d}\int_0^t \left(1-\frac{s}{t}\right)^{-c}\left(\frac{s}{t}\right)^{-d}\ ds.$$

Making the change of variables $\sigma = s/t$ and assuming that $bt \le 1$, we have

$$\begin{aligned}
t^{-c-d}\int_0^t (1-\frac{s}{t})^{-c}(\frac{s}{t})^{-d}\ ds &\le t^{1-c-d}\int_0^1 (1-\sigma)^{-c}\sigma^{-d}\ d\sigma \\
&= t^{1-c-d}\int_0^1 (1-\sigma)^{(1-c)-1}\sigma^{(1-d)-1}\ d\sigma \\
&= \mathcal{B}(1-c,1-d)(t\wedge b^{-1})^{1-c-d},
\end{aligned}$$

where $\mathcal{B}$ is given by (2.118).

On the other hand, if $bt > 1$, observe that

$$\begin{aligned}
\int_0^t \frac{e^{-b(t-s)}}{(t-s)^c(s\wedge b^{-1})^d}\ ds &= b^d\int_0^t (t-s)^{-c}e^{-b(t-s)}\ ds \\
&= b^d\int_0^t (t-s)^{-c}e^{-b(t-s)}\ ds \\
&= b^{d-1}\frac{1}{b^{-c}}\int_0^{bt} \sigma^{-c}e^{-\sigma}\ d\sigma \\
&\le (b^{-1})^{1-c-d}\int_0^{\infty} \sigma^{(1-c)-1}e^{-\sigma}\ d\sigma \\
&= \Gamma(1-c)(t\wedge b^{-1})^{1-c-d}.
\end{aligned}$$

□

Proposition 25. Let $n > 1$. Suppose that f is time-independent and satisfies $f = P_{\bar{\kappa}} f$. Let λ_f be given such that

$$\sup_{|y|\le\lambda_f} \|f(\,\cdot + iy)\|_{L^2} < \infty, \tag{2.151}$$

and $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ satisfy $\lambda(s) \leq \lambda_f$ whenever $0 \leq s \leq \tau$, for some $\tau > 0$. Then

$$M_f \sim_{\sigma,\bar{\kappa},\lambda_f,\tau} G, \tag{2.152}$$

where the constants are explicitly identified in (2.155) and (2.156).

Proof of Proposition 25. Let $z = x + iy$ with $x \in [0, L]^n$ and $|y| \leq \lambda(s)$. Then we can write $f(z) = \sum_{|k|\leq\bar{\kappa}/\kappa_0} \hat{f}(k)e^{i\kappa_0 k\cdot z}$. Observe that since $\kappa_0 = 2\pi/L$

$$\begin{aligned}\|f(\,\cdot + iy)\|_{L^2}^2 &= \sum_{|k|,|\ell|\leq\bar{\kappa}/\kappa_0} \hat{f}(k)\overline{\hat{f}(\ell)}e^{\kappa_0(k+\ell)\cdot y} \int_{[0,L]^n} e^{i\kappa_0(k-\ell)\cdot x}\, dx \\ &= (2\pi)^n \kappa_0^{-n} \sum_{|k|\leq\bar{\kappa}/\kappa_0} |\hat{f}(k)|^2 e^{2\kappa_0 k\cdot y}.\end{aligned}$$

This implies that

$$e^{-2\bar{\kappa}\lambda_f}\kappa_0^{-n/2}\|e^{\lambda(s)A^{1/2}}f\|_{\ell^2} \lesssim \|f(\,\cdot + iy)\|_{L^2} \lesssim \kappa_0^{-n/2}\|e^{\lambda(s)A^{1/2}}f\|_{\ell^2},$$

for all $|y| \leq \lambda(s)$. Hence

$$\frac{1}{\nu^2\kappa_0^3}\|e^{\lambda(s)A^{1/2}}f\|_{\ell^2} \sim_{\bar{\kappa},\lambda_f} \frac{\kappa_0^{n/2}}{\nu^2\kappa_0^3} \sup_{|y|\leq\lambda(s)} \|f(\,\cdot + iy)\|_{L^2}.$$

Now recall the following elementary facts:

- $\|f\|_{\ell^q} \leq \|f\|_{\ell^p} \lesssim_{p,q,\bar{\kappa}} \|f\|_{\ell^q}$ for $1 \leq p < q < \infty$;
- $\|f\|_{\ell^p} \leq \kappa_0^{-\sigma}\|f\|_\sigma \leq \left(\frac{\bar{\kappa}}{\kappa_0}\right)^\sigma \|f\|_{\ell^p}$ for $1 \leq p \leq \infty$

These imply that

$$\frac{\kappa_0^{-\sigma}}{\nu^2\kappa_0^3}\|f\|_{\lambda(s),\sigma} \sim_{\sigma,\bar{\kappa},\lambda_f} \frac{\kappa_0^{n/2}}{\nu^2\kappa_0^3}\|f(\,\cdot + iy)\|_{L^2}, \tag{2.153}$$

for all $|y| \leq \lambda(s)$. Obviously, if we set $y = 0$, then by the definition of the Grashof number (see (2.67)), we get

$$\frac{\kappa_0^{-\sigma}}{\nu^2\kappa_0^3}\sup_{0\le s\le\tau}\|f\|_{\lambda(s),\sigma}\sim_{\sigma,\bar\kappa,\lambda_f} G.$$

On the other hand, for $1\le q<\infty$, if we take the $L^q((0,\tau),ds/(\nu\kappa_0^2)^{-1})$ norm of (2.153), then

$$M_f\sim_{\sigma,\bar\kappa,\lambda_f,\tau}\frac{\kappa_0^{n/2}}{\nu^2\kappa_0^3}\|f(\,\cdot+iy)\|_{L^2}, \tag{2.154}$$

for all $|y|\le\lambda(s)$. Thus, by setting $y=0$ in (2.154) and by definition of (2.20), we deduce that

$$M_f\sim_{\sigma,\bar\kappa,\lambda_f,\tau} G.$$

In particular, we have

$$C_{\lambda_f,\bar\kappa,n}M_f\le(\nu\kappa_0^2\tau)^{1/q}G\le C_nM_f, \tag{2.155}$$

where $C_n:=(2\pi)^n$ and

$$C_{\lambda_f,\bar\kappa,n}:=(2\pi)^{-n}\left(\sum_{|k|\le\bar\kappa}1\right)^{-1/2}e^{-2\lambda_f\bar\kappa}\left(\frac{\kappa_0}{\bar\kappa}\right)^{\sigma}. \tag{2.156}$$

□

CHAPTER 3

CRITICAL AND SUPERCRITICAL SURFACE QUASI-GEOSTROPHIC EQUATION

3.1 PRELIMINARIES

We consider the two-dimensional dissipative surface quasi-geostrophic (SQG) equation given by

$$\begin{cases} \partial_t\theta + \Lambda^\kappa\theta - u\cdot\nabla\theta = 0, \\ u = (-R_2\theta, R_1\theta), \\ \theta(x,0) = \theta_0(x), \end{cases} \tag{3.1}$$

where R_j is the j-th Riesz transform, and $\Lambda^\kappa := (-\Delta)^{\kappa/2}$ for $0 < \kappa \leq 2$.

3.1.1 LITTLEWOOD-PALEY DECOMPOSITION AND RELATED INEQUALITIES

Let ψ_0 be a radial bump function such that $\psi_0(\xi) = 1$ when $[\|\xi\| \leq 1/2] \subset \mathbb{R}^d$, and

$$0 \leq \psi_0 \leq 1 \text{ and } \operatorname{spt}\psi_0 = [\|\xi\| \leq 1].$$

Define $\phi_0(\xi) := \psi_0(\xi/2) - \psi_0(\xi)$. Observe that

$$0 \leq \varphi_0 \leq 1 \text{ and } \operatorname{spt} \phi_0 = [2^{-1} \leq \|\xi\| \leq 2].$$

Now for each $j \in \mathbb{Z}$, define $\psi_j := (\psi_0)_{2^{-j}}$ and $\varphi_j := (\varphi_0)_{2^{-j}}$, where we use the notation

$$f_\lambda(x) := f(\lambda x). \tag{3.2}$$

for any $\lambda \geq 0$. Then obviously $\varphi_0 := \psi_1 - \psi_0$ and $\psi_{j+1} = \psi_j + \varphi_j$, so that

$$\operatorname{spt} \psi_j = [\|\xi\| \leq 2^{j-1}] \text{ and } \operatorname{spt} \varphi_j = [2^{j-1} \leq \|\xi\| \leq 2^{j+1}]. \tag{3.3}$$

Moreover, we have

$$\sum_{j \in \mathbb{Z}} \varphi_j(\xi) = 1, \text{ for } \xi \in \mathbb{R}^d \setminus \{\mathbf{0}\}.$$

One can then define

$$\begin{aligned} \triangle_k f &:= \check{\varphi}_k * f, \\ \breve{\triangle}_k f &:= \sum_{|k-\ell| \leq 2} \triangle_\ell f, \\ S_k f &:= \sum_{\ell \leq k-3} \triangle_\ell f. \end{aligned}$$

We call the operators $\triangle_k$ Littlewood-Paley blocks. For convenience, we will sometimes use the shorthand $f_k := \triangle_k f$.

For functions which are spectrally supported in a compact set, one has the Bernstein inequalities (cf. [5]), which we will invoke copiously throughout the article. We state it here in terms of Littlewood-Paley blocks. Note that we will use the following convention throughout the paper.

Notation. *$A \lesssim B$ to denote the relation $A \leq cB$ for some absolute constant $c > 0$. In our estimates, the constant c may change line to line, but will nevertheless remain an absolute constant.*

Lemma 26 (Bernstein inequalities)**.** Let $1 \leq p \leq q \leq \infty$ and $f \in \mathcal{S}'(\mathbb{R}^d)$. Then

$$2^{js}\|\triangle_j f\|_{L^q} \lesssim \|\Lambda^s \triangle_j f\|_{L^q} \lesssim 2^{js+d(1/p-1/q)}\|\triangle_j f\|_{L^p}, \tag{3.4}$$

for each $j \in \mathbb{Z}$ and $s \in \mathbb{R}$.

Since we will be working with L^p norms, we will also require the generalized Bernstein inequalities, which was proved in [14] and [76].

Lemma 27 (Generalized Bernstein inequalities)**.** Let $2 \leq p \leq \infty$ and $f \in \mathcal{S}'(\mathbb{R}^d)$. Then

$$2^{\frac{2sj}{p}}\|\triangle_j f\|_{L^p} \lesssim \|\Lambda^s |\triangle_j f|^{p/2}\|_{L^2}^{\frac{2}{p}} \lesssim 2^{\frac{2sj}{p}}\|\triangle_j f\|_{L^p}, \tag{3.5}$$

for each $j \in \mathbb{Z}$ and $s \in [0, 1]$.

In order to apply these inequalities, we will first need the following positivity lemma, which was initially proved in [25], and generalized by Ju in [53] (see also [16], [22]).

Lemma 28 (Positivity lemma)**.** Let $2 \leq p \leq \infty$, $f, \Lambda^s f \in L^p(\mathbb{R}^2)$. Then

$$\int \Lambda^s f |f|^{p-2} f \; dx \geq \frac{2}{p} \int (\Lambda^{\frac{s}{2}} |f|^{\frac{p}{2}})^2 \; dx. \tag{3.6}$$

We will also make use of the following heat kernel estimate, which was proved in [68] for L^2. We extend it to L^p.

Lemma 29. Let $2 \leq p < \infty$. Then there exist constants $c_1, c_2 > 0$ such that

$$e^{-c_1 t 2^{\kappa j}} \|\triangle_j u\|_{L^p} \leq \|e^{-t\Lambda^\kappa} \triangle_j u\|_{L^p} \leq e^{-c_2 t 2^{\kappa j}} \|\triangle_j u\|_{L^p}, \tag{3.7}$$

holds for all $t > 0$.

Proof. Let $u_j := e^{-t\Lambda^\kappa} \triangle_j u$. Then u_j satisfies the initial value problem

$$\begin{cases} \partial_t u_j + \Lambda^\kappa u_j = 0 \\ u_j(x, 0) = \triangle_j u(x). \end{cases} \tag{3.8}$$

Multipying (3.8) by $u_j |u_j|^{p-2}$ and integrating gives

$$\frac{1}{p} \frac{d}{dt} \|u_j\|_{L^p}^p + \int (\Lambda^\kappa u_j) u_j |u_j|^{p-2} \, dx = 0.$$

By applying Lemmas 27 and 28, then dividing by $\|u_j\|_{L^p}^{p-1}$ we obtain

$$\frac{d}{dt} \|u_j\|_{L^p} + c_1 2^{\kappa j} \|u_j\|_{L^p} \leq 0,$$

Similarly, by Hölder's inequality we obtain

$$\frac{d}{dt} \|u_j\|_{L^p} + c_2 2^{\kappa j} \|u_j\|_{L^p} \geq 0.$$

An application of Gronwall's inequality gives

$$e^{-c_2 2^{\kappa j} t} \|u_j(0)\|_{L^p} \leq \|u_j(t)\|_{L^p} \leq e^{-c_1 2^{\kappa j} t} \|u_j(0)\|_{L^p}, \tag{3.9}$$

which completes the proof. $\square$

3.1.2 BESOV SPACES

Let $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$. The *inhomogeneous Besov space* $B^s_{p,q}$ is the space defined by

$$B^s_{p,q} := \{f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{B^s_{p,q}} < \infty\}, \tag{3.10}$$

where, $\mathcal{S}'$ denotes the space of tempered distributions, and one can define the norm by

$$\|f\|_{B^s_{p,q}} := \|\check{\psi}_0 * f\|_{L^p} + \left(\sum_{j \geq 0} 2^{jsq} \|\triangle_j f\|^q_{L^p}\right)^{1/q}, \tag{3.11}$$

provided that $q < \infty$. The *homogeneous Besov space* $\dot{B}^s_{p,q}$ is the space defined by

$$\dot{B}^s_{p,q} := \{f \in \mathcal{Z}'(\mathbb{R}^d) : \|f\|_{\dot{B}^s_{p,q}} < \infty\}, \tag{3.12}$$

where $\mathcal{Z}'(\mathbb{R}^d)$ denotes the dual space of $\mathcal{Z}(\mathbb{R}^d) := \{f \in \mathcal{S}(\mathbb{R}^d) : \partial^\beta \hat{f}(0) = 0, \forall \beta \in \mathbb{N}^d\}$, and for $q < \infty$, the (semi)norm is given by

$$\|f\|_{\dot{B}^s_{p,q}} := \left(\sum_{j \in \mathbb{Z}} 2^{jsq} \|\triangle_j f\|^q_{L^p}\right)^{1/q}, \tag{3.13}$$

One then makes the usual modification for $q = \infty$. For more details, see [5] or [72].

3.1.3 GEVREY OPERATOR AND RELATED SPACES

Let $0 < \alpha \leq 1$ and $\gamma > 0$. We denote the *Gevrey operator* by the linear multiplier operator $T_{G_\gamma} = \mathscr{F}^{-1} G_\gamma \mathscr{F}$ where

$$G_\gamma(\xi) := \exp(\gamma \|\xi\|^\alpha), \tag{3.14}$$

and where $\|\cdot\|$ denotes the two-dimensional Euclidean norm. Note that this notation is not to be confused with f_λ as defined in (3.2). The meaning of this notation, however, will be clear from the context. For convenience, we write the multiplier operator, $T_{G_\gamma} f$, simply as

$$G_\gamma f \text{ or } \widetilde{f}. \tag{3.15}$$

We say that a function f is *Gevrey regular* if

$$\|G_\gamma f\|_{\dot{B}^s_{p,q}} < \infty, \tag{3.16}$$

for some $s \in \mathbb{R}$, $\gamma > 0$, and $1 \leq p, q \leq \infty$. Note that when $p = q = 2$, by the Bernstein inequalities one essentially recovers the usual definition of Gevrey regularity (cf. [66]), except for homogeneous Sobolev spaces. An important property of Gevrey regular functions is that estimates on higher-order derivatives follow immediately. In particular, it is elementary to show that functions which satisfy (3.16) automatically satisfy, for any $k > 0$, the estimate

$$\|D^k f\|_{\dot{B}^s_{p,q}} \leq C^k \frac{k^{k/\alpha}}{(\gamma\alpha)^{k/\alpha}} \|G_\gamma f\|_{\dot{B}^s_{p,q}}, \tag{3.17}$$

for some absolute constant $C > 0$. Indeed, this is one of the main reasons for working with Gevrey norms.

We will show that solutions of (3.1), whose initial values satisfy $\theta_0 \in B^{1+2/p-\kappa}_{p,q}$, are Gevrey regular up to some time $T > 0$, and in particular, belong to the space

$$X_T := \{v \in C((0,T); \dot{B}^{1+2/p-\kappa+\beta}_{p,q}(\mathbb{R}^2)) : \|v\|_{X_T} < \infty\}, \tag{3.18}$$

where T is possibly infinite, $2 \le p < \infty$, $1 \le q \le \infty$, $0 < \kappa \le 1$, and

$$\|v\|_{X_T} := \sup_{0<t<T} t^{\beta/\kappa} \|G_\gamma v(\,\cdot\,,t)\|_{\dot{B}^{1+2/p-\kappa+\beta}_{p,q}}, \tag{3.19}$$

where $\gamma := \lambda t^{\alpha/\kappa}$ for some $\alpha, \beta, \lambda > 0$.

3.2 MAIN RESULTS

Theorem 30. Let $2 \le p < \infty$ and $1 \le q \le \infty$. Let X_T be the space defined by (3.18) and (3.19). Suppose $\theta_0 \in B^{\sigma}_{p,q}(\mathbb{R}^2)$, where $\sigma := 1 + 2/p - \kappa$. Then there exists $T^* > 0$ and $\theta \in C([0,T^*); B^{\sigma}_{p,q}(\mathbb{R}^2))$ such that θ satisfies (3.1) and

$$\|\theta(\,\cdot\,)\|_{X_{T^*}} \lesssim \|\theta_0\|_{\dot{B}^{\sigma}_{p,q}}, \tag{3.20}$$

for some $0 \le \beta < \min\{\alpha, \kappa/2\}$ and $0 < \alpha < \kappa$. Moreover, there exists $C > 0$ such that if $\|\theta_0\|_{\dot{B}^{\sigma}_{p,q}} \le C$, then $T^* = \infty$.

Remark 1. *It will be clear from the proof that α can be chosen arbitrarily close to κ (see (3.100)).*

We also note that by following the proof of Theorem 30 one can actually prove a priori bounds on the approximating sequence in a stronger class Z_T, replacing X_T, where Z_T is defined as follows. First, define the space Y_T to be

$$Y_T := \{v \in C([0,T); \dot{B}^{1+2/p-\kappa}_{p,q}(\mathbb{R}^2)) : \|v\|_{Y_T} < \infty\}, \tag{3.21}$$

where

$$\|v\|_{Y_T} := \sup_{0\le t<T} \|G_\gamma v(\,\cdot\,,t)\|_{\dot{B}^{1+2/p-\kappa+\beta}_{p,q}}. \tag{3.22}$$

Then define Z_T by

$$Z_T := \{v \in C([0,T); \dot{B}^{1+2/p-\kappa}_{p,q}(\mathbb{R}^2)) : \|v\|_{Z_T} < \infty\}, \tag{3.23}$$

where the norm is defined by

$$\|v\|_{Z_T} := \max\{\|v\|_{X_T}, \|v\|_{Y_T}\}. \tag{3.24}$$

One is referred to Remark 45 for an outline of the proof.

This method is inspired by the work of Fujita and Kato in [45] and Weissler in [74], where the effect of instant regularization coming from the dissipation term is exploited to control the critical norm.

It immediately follows from Theorem 30, (3.17), and Stirling's approximation that the solutions of (3.1) with initial data belonging to $B^{1+2/p-\kappa}_{p,q}(\mathbb{R}^2)$ automatically satisfy certain higher-order decay estimates.

Corollary 31. Let $k > 1 + 2/p - \kappa$. Then the solution θ in Theorem 30 satisfies

$$\|D^k\theta(t)\|_{\dot{B}^{1+2/p-\kappa}_{p,q}} \lesssim C^k \frac{(k!)^{1/\alpha}}{t^{k/\kappa}} \|\theta_0\|_{\dot{B}^{1+2/p-\kappa}_{p,q}}, \tag{3.25}$$

for all $0 < t < T^*$, where $C := C(q, \alpha, \beta, \kappa)$.

It is well-documented (cf. [55], [68]) that in the presence of supercritical dissipation, product estimates are insufficient to control the nonlinearity in (3.1), and that commutators must be used instead to ensure that one remains in a perturbative regime. The proof of Theorem 30 will make use of the following commutator estimate for Gevrey regular functions, which is an extension of that found in Biswas (cf. [9]) to homogeneous Besov spaces. First let us recall the commutator bracket notation, $[A, B]$, which is defined as

$$[A, B] := AB - BA. \tag{3.26}$$

Theorem 32. Let $1 < p < \infty$ and $1 \leq q \leq \infty$. Let $\gamma, \delta > 0$ such that $\delta < 1$. Suppose $s, t \in \mathbb{R}$ satisfy the following

(i) $2/p < s < 1 + 2/p - \delta$,

(ii) $t < 2/p$,

(iii) $s + t > 2/p$.

Then there exists $C_j = C_j(\alpha, \delta, \gamma)$ such that

$$\|[G_\gamma \triangle_j, f]g\|_{L^p(\mathbb{R}^2)} \lesssim 2^{-(s+t-2/p)j} C_j \|G_\gamma f\|_{\dot{B}^s_{p,q}(\mathbb{R}^2)} \|G_\gamma g\|_{\dot{B}^t_{p,q}(\mathbb{R}^2)},$$

where

$$C_j := c_j \left(\gamma^{(\alpha-\delta)/\alpha} 2^{(\alpha-\delta)j} + 1 \right),$$

for some $(c_j)_{j\in\mathbb{Z}}$ such that $\|(c_j)\|_{\ell^q(\mathbb{Z})} \leq C$ for some absolute constant $C > 0$.

When one formally sets $\gamma = 0$, $p = 2$, and $\delta < \alpha$, Theorem 32 extends the commutator estimate of Miura (cf. [68]) to homogeneous Besov spaces.

Corollary 33. Suppose that p, q satisfy the conditions of Theorem 32 with $\delta = 0$. Then there exists $(c_j)_{j\in\mathbb{Z}} \in \ell^q$ such that

$$\|[\triangle_j, f]g\|_{L^p(\mathbb{R}^2)} \lesssim 2^{-(s+t-2/p)j} c_j \|f\|_{\dot{B}^s_{p,q}(\mathbb{R}^2)} \|g\|_{\dot{B}^t_{p,q}(\mathbb{R}^2)}.$$

This can be proved by closely following the proof of Theorem 32 and so we omit the details. The product estimate that corresponds to Theorem 32 is stated in the following theorem.

Theorem 34. Let $1 < p < \infty$ and $1 \leq q \leq \infty$. Suppose $s, t \in \mathbb{R}$ satisfy the following

(i) $s, t < 2/p$,

(ii) $s + t > 0$.

Then there exists $C > 0$ such that

$$\|G_\gamma(fg)\|_{\dot{B}^{s+t-2/p}_{p,q}(\mathbb{R}^2)} \leq C \|G_\gamma f\|_{\dot{B}^{s}_{p,q}(\mathbb{R}^2)} \|G_\gamma g\|_{\dot{B}^{t}_{p,q}(\mathbb{R}^2)}. \tag{3.27}$$

In order to prove Theorems 32 and 34, we apply the Bony paraproduct decomposition and view the resulting terms of both the commutator, $[G_\gamma \triangle_j, f]g$, and the product, $G_\gamma(fg)$, as bilinear multiplier operators, $T_m(f, g)$, which are written as

$$T_m(f, g) := \int\int e^{ix\cdot(\xi+\eta)} m(\xi, \eta) \hat{f}(\xi) \hat{g}(\eta) \; d\xi d\eta, \tag{3.28}$$

and show that for their corresponding symbols, m, the following estimate is satisfied for each multi-index β:

$$\left|\partial_\xi^{\beta_1} \partial_\eta^{\beta_2} m(\xi, \eta)\right| \lesssim_\beta \|\xi\|^{-|\beta_1|} \|\eta\|^{-|\beta_2|}. \tag{3.29}$$

In other words, we show that m is of Marcinkiewicz type. Note that condition (3.29) is weaker than that of Coifman-Meyer (cf. [15]). On the other hand, in general such multipliers need not map $L^p \times L^q$ into L^r *for any* $1 < p, q < \infty$ and $1/r = 1/p + 1/q$ (cf. [49]). This can be remedied by logarithmically strengthing (3.29) as Grafakos and Kalton demonstrated in [49]. In our case, however, the fact that we work with Besov spaces provides additional localizations which greatly simplify the situation.

Theorem 35. Suppose $m : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ satisfies (3.29) for sufficiently many multi-indices $|\beta| \geq 0$ with $\beta = \beta_1 + \beta_2$ and that for each fixed $\xi \in \mathbb{R}^d \setminus \{\mathbf{0}\}$, $m(\xi, \eta)$ is a smooth function of η with support contained in $[2^{j-1} \lesssim \|\eta\| \lesssim 2^{j+1}]$. Then for all $1 < p < \infty$, $1 \leq q \leq \infty$ such that $1/r = 1/p + 1/q$, the associated bilinear multiplier operator $T_m : L^p(\mathbb{R}^d) \times L^q(\mathbb{R}^d) \to L^r(\mathbb{R}^d)$ satisfies

$$\|T_m(f, g)\|_{L^r} \lesssim \|f\|_{L^p} \|g\|_{L^q}.$$

Remark 36. Note that the same conclusion holds with the roles of ξ, η and p, q reversed together in the above hypotheses.

A prototypical example of a bilinear operator satisfying (3.29) is $T(f, g) = Hf \cdot Hg$, where H is the Hilbert transform. Indeed, boundedness would then follow from Hölder's inequality. The role then of the smooth localization in η in Theorem 35 is that it essentially allows us to treat the bilinear multiplier as a product of linear ones, effectively reducing the situation to the simpler case of $Hf \cdot Hg$. Thus, Besov spaces provide an appropriate setting with which to work with bilinear Marcinkiewicz multipliers.

The proof of Theorem 35 is elementary and relies on classical techniques. We relegate its proof to the Appendix (Section 3.5), while the proofs of Theorems 30 and 32 will be given in Sections 3.4 and 3.3, respectively.

Remark 37. The notation T_m will be used to denote either a linear multiplier operator, $T_m f = \mathscr{F}^{-1}(m \mathscr{F} f)$, where $\mathscr{F}$ denotes the Fourier transform, or a bilinear multiplier operator $T_m(f, g)$, defined as in (3.28). However, it will be quite clear from the context which type of operator T_m is denoting.

3.3 COMMUTATOR ESTIMATES

In this section, we establish estimates for the product

$$G_\gamma \triangle_j (fg), \tag{3.30}$$

and for the commutator

$$[G_\gamma \triangle_j, f]g := G_\gamma \triangle_j (fg) - f G_\gamma \triangle_j g, \tag{3.31}$$

where $G_\gamma := e^{\gamma \Lambda^\alpha}$ and $0 < \alpha < \kappa \leq 1$, where κ is the order of dissipation in (3.1). For convenience, we will use the notation

$$\widetilde{f} := G_\gamma f. \tag{3.32}$$

To prove Theorems 32 and 34, we will require the Faà di Bruno formula, whose statement we recall from [5] for convenience. Note that by $\mathbb{N}$ and $\mathbb{N}^*$ we mean the set of positive integers with zero and the set $\mathbb{N} \setminus \{0\}$, respectively.

Lemma 38 (Faà di Bruno formula)**.** Let $u : \mathbb{R}^d \to \mathbb{R}^m$ and $F : \mathbb{R}^m \to \mathbb{R}$ be smooth functions. For each multi-index $\alpha \in \mathbb{N}^d$ with $|\alpha| > 0$ we have

$$\partial^\alpha (F \circ u) = \sum_{\mu,\nu} C_{\mu,\nu} \partial^\mu F \prod_{\substack{1 \leq |\beta| \leq |\alpha| \\ 1 \leq j \leq m}} (\partial^\beta u^j)^{\nu_{\beta_j}}, \tag{3.33}$$

where the coefficients $C_{\mu,\nu}$ are nonnegative integers, and the sum is taken over those μ and

ν such that $1 \leq |\mu|, |\nu| \leq |\alpha|$, $\nu_{\beta_j} \in \mathbb{N}^*$,

$$\sum_{1 \leq |\beta| \leq |\alpha|} \nu_{\beta_j} = \mu_j, \text{ for } 1 \leq j \leq m, \quad \text{and} \sum_{\substack{1 \leq |\beta| \leq |\alpha| \\ 1 \leq j \leq m}} \beta \nu_{\beta_j} = \alpha. \tag{3.34}$$

We will repeatedly apply this formula to functions of the form

$$(F \circ u)(\xi, \eta) = e^{\gamma R_{\alpha,\sigma}(\xi,\eta)},$$

where

$$R_{\alpha,\sigma}(\xi, \eta) := \|\xi + \sigma\eta\|^\alpha - \|\xi\|^\alpha - \|\eta\|^\alpha$$

or

$$R_{\alpha,\sigma}(\xi, \eta) := \|\xi\sigma + \eta\|^\alpha - \|\xi\|^\alpha - \|\eta\|^\alpha,$$

where $\sigma \in [0, 1]$. For convenience, we provide that application here. By Lemma 38 we have

$$\partial^\beta (F \circ u)(\xi, \eta) = \sum_{\mu,\nu} C_{\mu,\nu} \gamma^{|\mu|} e^{\gamma R_{\alpha,\sigma}(\xi,\eta)} \prod_{1 \leq |b| \leq |\beta|} (\partial^b R_{\alpha,\sigma}(\xi, \eta))^{\nu_b} \tag{3.35}$$

for all $\beta \in \mathbb{N}^2$, where $\nu = (\nu_1, \nu_2)$, $1 \leq |\mu| \leq |\beta|$ and

$$\sum_{1 \leq |b| \leq |\beta|} \nu_b = \mu \quad \text{and} \sum_{1 \leq |b| \leq |\beta|} b\nu_b = \beta. \tag{3.36}$$

Thus, in order to apply Theorem 35, we will require $R_{\alpha,\sigma}$ to satisfy certain derivative estimates.

Proposition 39. Let $0 < \alpha \leq 1$, $\sigma \in [0,1]$, and define $R_{\alpha,\sigma} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$R_{\alpha,\sigma}(\xi,\eta) := \|\xi + \eta\sigma\|^\alpha - \|\xi\|^\alpha - \|\eta\|^\alpha \tag{3.37}$$

Suppose that $\ell + 3 \leq k$ and $2^{k-1} \leq \|\xi\| \leq 2^{k+1}$ and $2^{\ell-1} \leq \|\eta\| \leq 2^{\ell+1}$. Then

$$\left|\partial_\xi^{\beta_1}\partial_\eta^{\beta_2} R_{\alpha,\sigma}(\xi,\eta)\right| \lesssim_{\beta,\alpha} 2^{\ell\alpha}\|\xi\|^{-|\beta_1|}\|\eta\|^{-|\beta_2|}, \tag{3.38}$$

for all multi-indices $\beta_1, \beta_2 \in \mathbb{N}^2$.

If $j + 3 \leq k$ with $2^{j-1} \leq \|\eta\| \leq 2^{j+1}$ and $2^{k-1} \leq \|\xi\|, \|\xi + \eta\| \leq 2^{k+1}$, then

$$\left|\partial_\xi^{\beta_1}\partial_\eta^{\beta_2} R_{\alpha,1}(\xi, -\xi - \eta)\right| \lesssim_{\beta,\alpha} 2^{k\alpha}\|\xi\|^{-|\beta_1|}\|\eta\|^{-|\beta_2|}, \tag{3.39}$$

for all $\beta_1, \beta_2 \in \mathbb{N}^d$.

Remark 40. If $R_{\alpha,\sigma}$ is given instead by

$$R_{\alpha,\sigma}(\xi,\eta) := \|\xi\sigma + \eta\|^\alpha - \|\xi\|^\alpha - \|\eta\|^\alpha, \tag{3.40}$$

then (3.38) and (3.39) all hold with the roles of k and ℓ reversed.

Proof. We prove (3.38). The inequality (3.39) can be obtained by direct estimation of derivatives.

For convenience we suppose $R_{\alpha,\sigma}$ is given by

$$R_{\alpha,\sigma}(\xi,\eta) := \|\xi + \eta\sigma\|^\alpha - \|\xi\|^\alpha - \|\eta\|^\alpha.$$

Let $\beta \in \mathbb{N}^4 \times \mathbb{N}^4$, where $\beta = (\beta_1, \beta_2) = (\beta_\xi, \beta_\eta)$, $\beta_j = (\beta_j^\xi, \beta_j^\eta)$, $\beta_j^\xi, \beta_j^\eta \in \mathbb{N}^2$ for $j = 1, 2$, and $\beta_\xi = \beta_1^\xi + \beta_2^\xi$ and $\beta_\eta = \beta_1^\eta + \beta_2^\eta$. Firstly, from the triangle inequality

$$|R_{\alpha,\sigma}(\xi,\eta)| \lesssim (1-\sigma)\|\eta\|^\alpha \lesssim 2^{\ell\alpha}$$

This proves (3.38) for $|\beta| = 0$. For $|\beta| \neq 0$, we apply the mean value theorem to write

$$R_{\alpha,\sigma}(\xi,\eta) = \int_0^1 \|\xi + \eta\tau\sigma\|^{\alpha-2}((\xi\cdot\eta)\sigma + \|\eta\|^2\sigma^2\tau)\ d\tau - \|\eta\|^\alpha.$$

Then observe that

$$\left|\partial^\beta R_\alpha(\xi,\eta)\right| \lesssim \sum_{\beta=\beta_1+\beta_2} c_\beta \int_0^1 \left(\|\xi + \eta\sigma\tau\|^{\alpha-2-|\beta_1|}\partial^{\beta_2}((\xi\cdot\eta)\sigma + \|\eta\|^2\sigma^2\tau)\right)\ d\tau + N_\alpha(\beta,\eta),$$

where $N_\alpha(\beta,\eta) = 0$ if $|\beta_j^\xi| \neq 0$ for some j, and $N_\alpha(\beta,\eta) = \|\eta\|^{\alpha-|\beta|}$ otherwise. Next observe that since $k \geq \ell + 3$, $\sigma,\tau \in [0,1]$, and $\xi \sim 2^k$, $\eta \sim 2^\ell$, we have

$$\|\xi + \eta\sigma\tau\| \gtrsim 2^k \gtrsim 2^\ell. \tag{3.41}$$

We also have

$$\left|\partial^{\beta_2}((\xi\cdot\eta)\sigma + \|\eta\|^2\sigma^2\tau)\right| \lesssim \begin{cases} 2^{k+\ell} & ,|\beta_2| = 0 \\ 2^k & ,|\beta_2| = |\beta_2^\eta| = 1 \\ 2^\ell & ,|\beta_2| = |\beta_2^\xi| = 1 \\ 1 & ,|\beta_2| = 2 \text{ and } |\beta_2^\xi| < 2 \\ 0 & ,|\beta_2| \geq 3 \text{ or } |\beta_2^\xi| = 2. \end{cases} \tag{3.42}$$

Now we consider three cases. First suppose that $|\beta_1| = 0, |\beta_2| \neq 0$. Using (3.41) and the fact that $\alpha < 1$, observe that

$$\|\xi + \eta\sigma\tau\|^{\alpha-2} \lesssim \begin{cases} 2^{\ell(\alpha-1)}2^{-k} & , |\beta_2| = 1 \text{ or } |\beta_2^\eta| = |\beta_2^\xi| = 1 \\ 2^{\ell(\alpha-2)} & , |\beta_2| = |\beta_2^\eta| = 2. \end{cases} \tag{3.43}$$

Thus, combining (3.42) and (3.43) gives

$$\left|\partial^\beta R_{\alpha,\sigma}(\xi,\eta)\right| \lesssim 2^{\ell\alpha}2^{-k|\beta_2^\xi|}2^{-\ell|\beta_2^\eta|},$$

which implies (3.38) since $\beta = (0, 0, \beta_2^\xi, \beta_2^\eta)$.

Now suppose $|\beta_1| \neq 0$ and $|\beta_2| = 0$. Applying (3.41) then gives

$$\|\xi + \eta\sigma\tau\|^{\alpha-2-|\beta_1|} \lesssim \begin{cases} 2^{\ell(\alpha-1)}2^{-k-|\beta_1^\xi|} & , |\beta_1^\eta| = 0 \neq |\beta_1^\xi| \\ 2^{\ell(\alpha-1-|\beta_1^\eta|)}2^{-k} & , |\beta_1^\xi| = 0 \neq |\beta_1^\eta| \\ 2^{\ell(\alpha-1-|\beta_1^\eta|)}2^{k(-1-|\beta_1^\xi|)} & , |\beta_1^\xi|, |\beta_1^\eta| \neq 0. \end{cases} \tag{3.44}$$

Thus, by combining (3.42) and (3.44) we get

$$\left|\partial^\beta R_{\alpha,\sigma}(\xi,\eta)\right| \lesssim 2^{\ell\alpha}2^{-k|\beta_1^\xi|}2^{-\ell|\beta_1^\eta|},$$

which again implies (3.38) since $\beta = (\beta_1^\xi, \beta_1^\eta, 0, 0)$.

Finally, if $\beta_1 \neq 0, \beta_2 \neq 0$, we may combine the argumentation of the previous two cases to obtain

$$\left|\partial^\beta R_{\sigma,\alpha}(\xi,\eta)\right| \lesssim 2^{\ell\alpha}2^{-k|\beta_\xi|}2^{-\ell|\beta_\eta|}, \tag{3.45}$$

This establishes (3.38) for all $\beta \in \mathbb{N}^4 \times \mathbb{N}^4$. □

We will also need the following "rotation" lemma.

Lemma 41. Let T_m be a bilinear multiplier operator with multiplier $m : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$. Then for $\widetilde{m}(\xi, \eta) := (-1)^d m(\xi, -\xi - \eta)$, we have

$$\langle T_m(f, g), h\rangle = \langle T_{\widetilde{m}}(h, g), f\rangle, \tag{3.46}$$

for all $f, g, h \in \mathcal{S}(\mathbb{R}^d)$. Moreover, if $T_m : L^p \times L^q \to L^r$ is bounded for some $1/r = 1/p + 1/q$, then $T_{\widetilde{m}} : L^{r'} \times L^q \to L^{p'}$ is bounded, where p', r' are the Hölder conjugates of p, r, respectively.

Proof. By change of variables we have

$$\begin{aligned}
\int T_m(f, g)(x)h(x)\ dx &= \int\int\int e^{ix\cdot(\xi+\eta)} m(\xi, \eta)\hat{f}(\xi)\hat{g}(\eta)h(x)\ d\xi\ d\eta\ dx \\
&= (-1)^d \int\int\int e^{-ix\cdot\nu} m(\xi, \xi - \nu)\hat{f}(\xi)\hat{g}(-\nu - \xi)h(x)\ dx\ d\xi\ d\nu \\
&= (-1)^d \int\int m(\xi, -\nu - \xi)\hat{g}(-\xi - \nu)\hat{h}(\nu)\hat{f}(\xi)\ d\nu\ d\xi \\
&= (-1)^d \int\int\int e^{-ix\cdot\xi} m(\xi, -\xi - \nu)\hat{g}(-\nu - \xi)\hat{h}(\nu)f(x)\ d\nu\ d\xi\ dx \\
&= \langle T_{\widetilde{m}}(h, g), f\rangle,
\end{aligned}$$

as desired. Boundedness of $T_{\widetilde{m}}$ then follows from duality. □

Remark 42. Observe that if $1 < p, r < \infty$, then $1 < p', r' < \infty$ as well. Therefore, if T_m is bounded in the range $1/r = 1/p + 1/q$ for $1 < p, r < \infty$, then $T_{\widetilde{m}}$ is also bounded in the same range.

We will first prove Theorem 34 since the estimates there will be used to prove Theorem

32. As a preliminary, we recall the paraproduct decomposition:

$$fg = \sum_k S_k f \triangle_k g + \sum_k \triangle_k f S_k g + \sum_k \breve{\triangle}_k f \triangle_k g. \tag{3.47}$$

This implies that

$$\begin{aligned}[G_\gamma \triangle_j, f]g = &\sum_k G_\gamma \triangle_j (S_k f \triangle_k g) + G_\gamma \triangle_j (\triangle_k f S_k g) + G_\gamma \triangle_j (\breve{\triangle}_k f \triangle_k g) \\ &- \left(\sum_k (S_k f)(\triangle_j \triangle_k \widetilde{g}) + (\triangle_k f)(\triangle_j S_k \widetilde{g}) + (\breve{\triangle}_k f)(\triangle_j \triangle_k \widetilde{g}) \right).\end{aligned} \tag{3.48}$$

Then by the localization properties in (3.3), we can reduce (3.48) to

$$\begin{aligned}[G_\gamma \triangle_j, f]g = &\sum_{|k-j|\le 4} \Big\{ [G_\gamma \triangle_j, S_k f] \triangle_k g + G_\gamma \triangle_j (\triangle_k f S_k g) + G_\gamma \triangle_j (\breve{\triangle}_k f \triangle_k g) \Big\} \\ &+ \sum_{k \ge j+5} G_\gamma \triangle_j (\breve{\triangle}_k f \triangle_k g) \\ &- \sum_{k \ge j+1} \triangle_k f \triangle_j S_k \widetilde{g} - \sum_{|k-j|\le 2} \breve{\triangle}_k f \triangle_j \triangle_k \widetilde{g}.\end{aligned} \tag{3.49}$$

3.3.1 PROOF OF THEOREM 34

Observe that $G_\gamma \triangle_j (fg)$ is precisely the first line of (3.48). By symmetry and localization, it suffices to consider only

$$\sum_{|k-j|\le 4} \Big[G_\gamma \triangle_j (\triangle_k f S_k g) + G_\gamma \triangle_j (\breve{\triangle}_k f \triangle_k g) \Big] \text{ and } \sum_{k \ge j+5} G_\gamma \triangle_j (\breve{\triangle}_k f \triangle_k g)$$

CASE: $k \ge j + 5$

First, we rewrite $G_\gamma \triangle_j (\breve{\triangle}_k f \triangle_k g)$ as

$$G_\gamma \triangle_j (G_\gamma^{-1} \breve{\triangle}_k \widetilde{f} G_\gamma^{-1} \triangle_k \widetilde{g}) \tag{3.50}$$

The multiplier associated to (3.50) is

$$m_{k,j}(\xi,\eta) := e^{\gamma(\|\xi+\eta\|^{\alpha}-\|\xi\|^{\alpha}-\|\eta\|^{\alpha})}\varphi_j(\xi+\eta)\breve{\varphi}_k(\xi)\varphi_k(\eta), \tag{3.51}$$

where $\breve{\varphi}_k = \sum_{|k-\ell|\le 2}\varphi_\ell$. By Lemma 41, in order to apply Theorem 35, it suffices to prove

$$|\partial_\xi^{\beta_1}\partial_\eta^{\beta_2} m'_{k,j}(\xi,\eta)| \lesssim \|\xi\|^{-|\beta_1|}\|\eta\|^{-|\beta_2|}, \tag{3.52}$$

where $m'_{k,j}(\xi,\eta) = m_{k,j}(\xi,-\xi-\eta)$. Once the required L^p bounds are deduced, we then show that the obtained estimate is summable in ℓ^q with respect to j.

So first observe that for $\beta = (\beta_1,\beta_2)$, by (3.35), (3.36), and (3.39) we have

$$\begin{aligned}
&|\partial^\beta m_{k,j}(\xi,-\xi-\eta)| \\
&\quad\lesssim \sum_{\mu,\nu} C_{\mu,\nu}\gamma^{|\mu|}e^{\gamma(\|\eta\|^{\alpha}-\|\xi\|^{\alpha}-\|\xi+\eta\|^{\alpha})}\prod_{1\le|b|\le|\beta|}(2^{k(\alpha-|b|)})^{\nu_b} \\
&\quad\lesssim 2^{-k|\beta|}\sum_{\mu,\nu} C_{\mu,\nu}(\gamma 2^{k\alpha})^{|\mu|}e^{\gamma(\|\eta\|^{\alpha}-\|\xi\|^{\alpha}-\|\xi+\eta\|^{\alpha})}
\end{aligned}$$

Also, by the triangle inequality

$$\|\eta\|^{\alpha}-\|\xi\|^{\alpha}-\|\xi+\eta\|^{\alpha} \lesssim -c_\alpha 2^{k\alpha}.$$

for some absolute constant $c_\alpha > 0$. Thus

$$|\partial^\beta m_{k,j}(\xi,-\xi-\eta)| \qquad \lesssim 2^{-k|\beta|}\sum_{\mu,\nu} C_{\mu,\nu}(\gamma 2^{k\alpha})^{|\mu|}e^{-c_\alpha\gamma 2^{k\alpha}} \lesssim 2^{-k|\beta_1|}2^{-k|\beta_2|} \tag{3.53}$$

holds for all $\xi \in \mathbb{R}^2$, which implies (3.52).

Now, let $\sigma = s + t - 2/p$. Observe that by the Bernstein and Theorem 35 we have

$$\begin{aligned}
\|G_\gamma \triangle_j(\breve{\triangle}_k f \triangle_k g)\|_{L^p} &\lesssim 2^{j(2/p)} \|G_\gamma \triangle_j(\breve{\triangle}_k f \triangle_k g)\|_{L^{p/2}} \\
&\lesssim 2^{j(2/p)} \|\breve{\triangle}_k \widetilde{f}\|_{L^p} \|\triangle_k \widetilde{g}\|_{L^p} \\
&\lesssim 2^{-\sigma j} \underbrace{2^{-(s+t)(k-j)}}_{a_{k-j}} \underbrace{(2^{sk}\|\breve{\triangle}_k \widetilde{f}\|_{L^p})}_{b_k} \underbrace{(2^{tk}\|\triangle_k \widetilde{g}\|_{L^p})}_{c_k}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\sum_{k \geq j+5} 2^{\sigma j} \|G_\gamma \triangle_j(\breve{\triangle}_k f \triangle_k g)\|_{L^p} &\lesssim \sum_k \chi_{[n\geq 5]}(k-j) a_{k-j} b_k c_k \\
&\lesssim \left(\sum_k \chi_{[n\geq 5]}(k-j) a_{k-j} b_k \right) \left(\sup_k c_k \right).
\end{aligned}$$

Observe that by Young's convolution inequality we have

$$\left(\sum_{k \geq j+5} (a_{k-j} b_k)^q \right)^{1/q} \leq \left(\sum_{k\geq 5} a_k \right) \left(\sum_k b_k^q \right)^{1/q},$$

which is finite provided that $s + t > 0$.

Therefore

$$2^{(s+t-2/p)j} \sum_{k\geq j+5} \|G_\gamma \triangle_j(\breve{\triangle}_k f \triangle_k g)\|_{L^r} \lesssim c_j \|\widetilde{f}\|_{\dot{B}^s_{p,q}} \|\widetilde{g}\|_{\dot{B}^t_{p,\infty}}, \tag{3.54}$$

where

$$c_j := \|\widetilde{f}\|^{-1}_{\dot{B}^s_{p,q}} \sum_{k\geq j+5} a_{k-j} b_k$$

and satisfies $(c_j)_{j\in\mathbb{Z}} \in \ell^q$. This finishes the case $k \geq j+5$ and hence, the proof of Theorem 34. □

CASE: $|k-j| \leq 4$

It suffices to consider $G_\gamma \triangle_j(\triangle_k f S_k g)$ since the term $G_\gamma \triangle_j(\breve{\triangle}_k f \triangle_k g)$ is easier.

First, let us we rewrite $G_\gamma \triangle_j(\triangle_k f S_k g)$ as

$$\sum_{\ell \leq k-3} G_\gamma \triangle_j(G_\gamma^{-1} \triangle_k \widetilde{f} G_\gamma^{-1} \triangle_\ell \widetilde{g}). \tag{3.55}$$

We claim that the associated multiplier satisfies the following bounds

$$|\partial_\xi^{\beta_1} \partial_\eta^{\beta_2} m_{j,k,\ell}(\xi,\eta)| \lesssim \|\xi\|^{-|\beta_1|} \|\eta\|^{-|\beta_2|}, \tag{3.56}$$

where

$$m_{j,k,\ell}(\xi,\eta) = e^{\gamma(\|\xi+\eta\|^\alpha - \|\xi\|^\alpha - \|\eta\|^\alpha)} \varphi_j(\xi+\eta)\varphi_k(\xi)\varphi_\ell(\eta). \tag{3.57}$$

To this end, let $\beta = (\beta_\xi, \beta_\eta)$ and observe that by (3.35) and Proposition 39 we have

$$|\partial^\beta e^{\gamma R_\alpha(\xi,\eta)}| \lesssim \sum_{\mu,\nu} C_{\mu,\nu} \gamma^{|\mu|} e^{\gamma R_\alpha(\xi,\eta)} \prod_{1 \leq |b| \leq |\beta|} (2^{\ell(\alpha - |b_\eta|)} 2^{-k|b_\xi|})^{\nu_b}.$$

Since $\|\eta\| \sim 2^\ell$ and $k - \ell \geq 3$, it follows by Lemma 46 that

$$\|\xi+\eta\|^\alpha - \|\xi\|^\alpha - \|\eta\|^\alpha \lesssim -c_\alpha 2^{\ell\alpha}.$$

Thus, by (3.36) we get

$$
\begin{aligned}
|\partial^\beta e^{\gamma R_\alpha(\xi,\eta)}| &\lesssim \sum_{\mu,\nu} C_{\mu,\nu}\gamma^{|\mu|}e^{-c_\alpha\gamma 2^{\ell\alpha}}2^{\ell(\alpha|\mu|-|\beta_\eta|)}2^{-k|\beta_\xi|} \\
&\lesssim 2^{-k|\beta_\xi|}2^{-\ell|\beta_\eta|}\sum_{\mu,\nu} C_{\mu,\nu}(\gamma 2^{\ell\alpha})^{|\mu|}e^{-c_\alpha\gamma 2^{\ell\alpha}} \\
&\lesssim 2^{-k|\beta_\xi|}2^{-\ell|\beta_\eta|}
\end{aligned}
\tag{3.58}
$$

holds for all $\xi \in \mathbb{R}^2$

Hence, by the product rule and the fact that $2^k \sim 2^j$, we can conclude that

$$
|\partial_\xi^{\beta_1}\partial_\eta^{\beta_2} m_{j,k}(\xi,\eta)| \lesssim 2^{-k|\beta_1|}2^{-\ell|\beta_2|}.
$$

for all $\xi \in \mathbb{R}^2$, which implies (3.56).

Therefore by Theorem 35, we have

$$
\|G_\gamma \triangle_j(\triangle_k f S_k g)\|_{L^r} \lesssim \sum_{\ell \le k-1} \|\triangle_k \widetilde{f}\|_{L^p}\|\triangle_\ell \widetilde{g}\|_{L^q},
\tag{3.59}
$$

where $1/r = 1/p + 1/q$ and $1 \le r < \infty$, $1 < p < \infty$, $1 < q \le \infty$.

Now let $\sigma = s + t - 2/p$ and $N > 1$. Let $p^* := (pN)/(N-1)$. Then by (3.59), the Bernstein inequalities, and the fact that $|k - j| \le 4$, we have

$$
\begin{aligned}
&2^{\sigma j}\|G_\gamma \triangle_j(\triangle_k f S_k g)\|_{L^p} \\
&\quad\lesssim \sum_{\ell \le k-1} 2^{(\sigma - s - t + 2/p^*)k}2^{sk}\|\triangle_k \widetilde{f}\|_{L^{p^*}}2^{t\ell}\|\triangle_\ell \widetilde{g}\|_{L^p}2^{-(2/p^*-t)(k-\ell)} \\
&\quad\lesssim 2^{sj}\|\triangle_j \widetilde{f}\|_{L^p}\sum_{\ell \le k-1} 2^{t\ell}\|\triangle_\ell \widetilde{g}\|_{L^p}2^{-(2/p^*-t)(k-\ell)}
\end{aligned}
$$

Let $t < 2/p$. Observe that for N large enough, we have $t < 2/p^*$. Then

$$2^{(s+t-2/p)j}\|G_\gamma \triangle_j(\triangle_k f S_k g)\|_{L^p} \lesssim C_j \|\widetilde{f}\|_{\dot{B}^s_{p,\infty}} \|\widetilde{g}\|_{\dot{B}^t_{p,q}}, \tag{3.60}$$

where

$$C_j := \sum_{\ell \le j+2} 2^{t\ell}\|\triangle_\ell \widetilde{g}\|_{L^p} 2^{(t-2/p^*)(j-\ell)},$$

which satisfies $(C_j)_{j\in\mathbb{Z}} \in \ell^q$. This establishes the case $|k-j| \le 4$.

3.3.2 PROOF OF THEOREM 32

CASES: $k \ge j+1$ **AND** $|k-j| \le 1$

The corresponding terms are $\triangle_k f \triangle_j S_k \widetilde{g}$ and $\triangle_k f \triangle_j \triangle_k \widetilde{g}$, respectively. By Hölder's inequality and Bernstein we have

$$2^{\sigma j}\|\triangle_k f \triangle_j S_k \widetilde{g}\|_{L^p} \lesssim c_j 2^{-(s-2/p)(k-j)} 2^{sk}\|\triangle_k f\|_{L^p}\|\widetilde{g}\|_{\dot{B}^t_{p,q}}. \tag{3.61}$$

where

$$c_j := \|\widetilde{g}\|^{-1}_{\dot{B}^t_{p,q}} 2^{tj}\|\triangle_j \widetilde{g}\|_{L^p}.$$

Observe that by Hölder's inequality

$$\sum_{k\ge j+1} 2^{-(s-2/p)(k-j)}\chi_{[n\ge 1]}(k-j)2^{sk}\|\triangle_k f\|_{L^p} \lesssim \left(\sum_{k\ge 1} 2^{-(s-2/p)kq'}\right)^{1/q'} \|f\|_{\dot{B}^s_{p,q}},$$

which is finite provided that $s - 2/p > 0$. Therefore

$$2^{(s+t-2/p)j} \sum_{k \geq j+1} \|\triangle_k f \triangle_j S_k \widetilde{g}\|_{L^p} \lesssim c_j \|f\|_{\dot{B}^s_{p,q}} \|\widetilde{g}\|_{\dot{B}^t_{p,q}} \tag{3.62}$$

Similarly, we have for any $s \in \mathbb{R}$

$$2^{(s+t-2/p)j} \sum_{|k-j| \leq 1} \|\breve{\triangle}_k f \triangle_k \widetilde{g}\|_{L^p} \lesssim c_j \|f\|_{\dot{B}^s_{p,q}} \|\widetilde{g}\|_{\dot{B}^t_{p,q}}, \tag{3.63}$$

where

$$c_j := \|\widetilde{g}\|^{-1}_{\dot{B}^t_{p,q}} 2^{tj} \|\triangle_j \widetilde{g}\|_{L^p}.$$

CASE: $k \geq j + 5$

The derivative estimates for the corresponding multiplier remain the same as those from Theorem 34, except that we sum over k differently since now it is assumed that $s+t-2/p > 0$.

Since (3.52) holds, we know that Theorem 35 implies

$$\|G_\gamma \triangle_j (\breve{\triangle}_k f \triangle_k g)\|_{L^p} \lesssim \|\breve{\triangle}_k \widetilde{f}\|_{L^p} \|\triangle_k \widetilde{g}\|_{L^\infty}.$$

Thus, for $\sigma = s + t - 2/p$, by the Bernstein inequalities we have that

$$\sum_{k \geq j+5} 2^{\sigma j} \|G_\gamma \triangle_j (\triangle_k f \triangle_k g)\|_{L^p}$$
$$\lesssim \sum_k \underbrace{\chi_{[n \geq 5]}(k-j)}_{\mu_{k-j}} \underbrace{2^{-(s+t-2/p)(k-j)}}_{a_{k-j}} \underbrace{2^{sk} \|\breve{\triangle}_k \widetilde{f}\|_{L^p}}_{b_k} \underbrace{2^{tk} \|\triangle_k \widetilde{g}\|_{L^p}}_{c_k}.$$

As before Young's convolution inequality implies

$$\left(\sum_k (\mu_{k-j}a_{k-j}b_k)^q\right)^{1/q} \le \left(\sum_{k\ge 5} a_k\right)\left(\sum_k b_k^q\right)^{1/q},$$

which will be finite provided that

$$s + t - 2/p > 0.$$

Thus

$$2^{(s+t-2/p)j}\sum_{k\ge j+5}\|G_\gamma\triangle_j(\breve{\triangle}_k f\triangle_k g)\|_{L^r} \lesssim c_j\|\widetilde{f}\|_{\dot{B}^s_{p,q}}\|\widetilde{g}\|_{\dot{B}^t_{p,\infty}}, \tag{3.64}$$

with c_j given by

$$c_j := \|\widetilde{f}\|^{-1}_{\dot{B}^s_{p,q}}\sum_k \mu_{k-j}a_{k-j}b_k.$$

CASE: $|k-j|\le 4$

From the proof of Theorem 34, it suffices to consider the commutator term, $[G_\gamma\triangle_j, S_k f]\triangle_k g$, which we view as $T_{m_{j,k}}(S_k f, \triangle_k g)$. Indeed, observe that

$$\begin{aligned}
&T_{m_{j,k}}(S_k f, \triangle_k g)(x)\\
&\qquad = \int\int e^{ix\cdot(\xi+\eta)}\left[G_\gamma(\xi+\eta)\varphi_j(\xi+\eta) - G_\gamma(\eta)\varphi_j(\eta)\right]\psi_k(\xi)\varphi_k(\eta)\hat{f}(\xi)\hat{g}(\eta)\; d\xi d\eta.
\end{aligned}$$

Then by the mean value theorem

$$T_{m_{j,k}}(S_k f, \triangle_k g)(x) = \sum_{i=1,2}\sum_{\ell\le k-3}\int_0^1 T_{m_{i,j,k,\ell,\sigma}}(\triangle_\ell\partial_i\widetilde{f}, \triangle_k\widetilde{g})(x)\; d\sigma,$$

where

$$m_{i,j,k,\ell,\sigma}(\xi,\eta) = m_A(\xi,\eta) + m_B(\xi,\eta),$$

and

$$\begin{aligned} m_A(\xi,\eta) &:= \alpha\gamma e^{\gamma R_{\alpha,\sigma}(\xi,\eta)} \|\xi\sigma+\eta\|^{\alpha-2}(\xi_i\sigma+\eta_i)\varphi_j(\xi\sigma+\eta)\varphi_\ell(\xi)\varphi_k(\eta) \\ m_B(\xi,\eta) &:= e^{\gamma R_{\alpha,\sigma}(\xi,\eta)}(\partial_i\varphi_0)(2^{-j}(\xi\sigma+\eta))2^{-j}\varphi_\ell(\xi)\varphi_k(\eta). \end{aligned}$$

Now observe that since $\|\xi\| \sim 2^\ell$, $\|\eta\| \sim 2^k$, and $k - \ell \geq 3$, by Lemma 46 there exists a constant $c_\alpha > 0$ such that

$$\|\xi\sigma+\eta\|^\alpha - \|\xi\sigma\|^\alpha - \|\eta\|^\alpha \leq -c_\alpha\|\xi\|^\alpha, \text{ for } \sigma \geq 1/2, \tag{3.65}$$

and by the triangle inequality

$$\|\xi\sigma+\eta\|^\alpha - \|\xi\|^\alpha - \|\eta\|^\alpha \leq -c'_\alpha\|\xi\|^\alpha, \text{ for } \sigma \leq 1/2. \tag{3.66}$$

This implies that

$$e^{\gamma R_{\alpha,\sigma}}(\xi,\eta) \lesssim \begin{cases} e^{-c'_\alpha\gamma\|\xi\|^\alpha}, & \sigma \leq 1/2 \\ e^{-c_\alpha\gamma\|\xi\|^\alpha}e^{-(1-\sigma^\alpha)\|\xi\|^\alpha}, & \sigma > 1/2 \end{cases}. \tag{3.67}$$

Suppose that $\sigma \leq 1/2$ and observe that by Proposition 39, Faà di Bruno, and (3.67), we have

$$
\begin{aligned}
|\partial^\beta e^{\gamma R_{\alpha,\sigma}(\xi,\eta)}| &\lesssim \sum_{\mu,\nu} C_{\mu,\nu}\gamma^{|\mu|} e^{\gamma R_{\alpha,\sigma}(\xi,\eta)} \prod_{1\le|b|\le|\beta|} (2^{\ell(\alpha-|b_\xi|)}2^{-k|b_\eta|})^{\nu_b} \\
&\lesssim 2^{-\ell|\beta_1|}2^{-k|\beta_2|} \sum_{\mu,\nu} C_{\mu,\nu}(\gamma 2^{\ell\alpha})^{|\mu|} e^{-c_\alpha \gamma 2^{\ell\alpha}} \\
&\lesssim e^{-(c'_\alpha/2)\gamma 2^{\ell\alpha}} \|\xi\|^{-|\beta_1|}\|\eta\|^{-|\beta_2|}. \qquad (3.68)
\end{aligned}
$$

Similarly, for $\sigma \geq 1/2$, using (3.67) we obtain

$$
\begin{aligned}
|\partial^\beta e^{\gamma R_{\alpha,\sigma}(\xi,\eta)}| &\lesssim \sum_{\mu,\nu} C_{\mu,\nu}\gamma^{|\mu|} e^{-c_\alpha\gamma 2^{\ell\alpha}} e^{-(1-\sigma^\alpha)\gamma\|\xi\|^\alpha} \prod_{1\le|b|\le|\beta|} (\partial^b R_{\alpha,\sigma}(\xi,\eta))^{\nu_b} \\
&\lesssim \sum_{\mu,\nu} C_{\mu,\nu}\gamma^{|\mu|} e^{-c_\alpha\gamma 2^{\ell\alpha}} \prod_{1\le|b|\le|\beta|} (2^{\ell(\alpha-|b_\xi|)}2^{-k|b_\eta|})^{\nu_b} \\
&\lesssim e^{-(c_\alpha/2)\gamma 2^{\ell\alpha}} \|\xi\|^{-|\beta_1|}\|\eta\|^{-|\beta_2|}. \qquad (3.69)
\end{aligned}
$$

For the other factors, observe that since $\|\xi\sigma+\eta\| \sim 2^j$ we have

$$
\left|\partial^\beta \|\xi\sigma+\eta\|^{\alpha-2}\right| \lesssim \|\xi\sigma+\eta\|^{\alpha-2-|\beta|} \lesssim 2^{j(\alpha-2)}\|\xi\|^{-|\beta_\xi|}\|\eta\|^{-|\beta_\eta|} \qquad (3.70)
$$

$$
\left|\partial^\beta(\xi_i\sigma+\eta_i)\right| \lesssim \begin{cases} 2^\ell + 2^k & , |\beta_\xi| = 0 \\ 1 & , |\beta| = 1 \text{ and } |\beta^i_\xi| \text{ or } |\beta^i_\eta| = 1 \\ 0 & , |\beta| \geq 2 \text{ or } |\beta^{i'}| \neq 0 \; i' \neq i \end{cases} \qquad (3.71)
$$

It follows from (3.70) and (3.71) that

$$
\left|\partial^\beta\left(\|\xi\sigma+\eta\|^{\alpha-2}(\xi_i\sigma+\eta_i)\right)\right| \lesssim 2^{j(\alpha-1)}2^{-\ell|\beta_\xi|}2^{-k|\beta_\eta|}. \qquad (3.72)
$$

We also have

$$\left|\partial_\xi^\beta \varphi_\ell(\xi)\right| \lesssim 2^{-\ell|\beta|} \lesssim \|\xi\|^{-|\beta|}, \tag{3.73}$$

$$\left|\partial_\eta^\beta \varphi_k(\eta)\right| \lesssim 2^{-k|\beta|} \lesssim \|\eta\|^{-|\beta|} \tag{3.74}$$

for all $\eta \in \mathbb{R}^2$.

Therefore, combining (3.68), (3.69) and (3.72)-(3.74), we can deduce that

$$\begin{aligned}\left|\partial_\xi^{\beta_1}\partial_\eta^{\beta_2} m_A(\xi,\eta)\right| &\lesssim \gamma 2^{-j(1-\alpha)} e^{-(c_\alpha/2)\gamma 2^{\ell\alpha}} \|\xi\|^{-|\beta_1|}\|\eta\|^{-|\beta_2|} \\ &\lesssim \gamma^{1-\delta/\alpha} 2^{-j(1-\alpha)} 2^{-\ell\delta} \|\xi\|^{-|\beta_1|}\|\eta\|^{-|\beta_2|}.,\end{aligned} \tag{3.75}$$

for any $\delta > 0$.

On the other hand, we can estimate m_B using (3.68) and (3.69) by

$$\left|\partial_\xi^{\beta_1}\partial_\eta^{\beta_2} m_B(\xi,\eta)\right| \lesssim 2^{-j} \|\xi\|^{-|\beta_1|}\|\eta\|^{-|\beta_2|}. \tag{3.76}$$

Fix $N > 1$ and let $p^* = (pN)/(N-1)$ with $(p^*)' = pN$ so that $1/p = 1/(p^*)' + 1/p*$. Then by Theorem 35 and the Bernstein inequalities

$$\|T_{m_A}(\triangle_\ell \partial_i \widetilde{f}, \triangle_k \widetilde{g})\|_{L^p} \lesssim \gamma^{1-\delta/\alpha} 2^{-j(1-\alpha)} 2^{(1-\delta)\ell} \|\triangle_\ell \widetilde{f}\|_{L^{(p^*)'}} \|\triangle_k \widetilde{g}\|_{L^{p^*}}, \tag{3.77}$$

$$\|T_{m_B}(\triangle_\ell \partial_i \widetilde{f}, \triangle_k \widetilde{g})\|_{L^p} \lesssim 2^{-j} 2^{\ell} \|\triangle_\ell \widetilde{f}\|_{L^{(p^*)'}} \|\triangle_k \widetilde{g}\|_{L^{p^*}}. \tag{3.78}$$

Suppose $s < 1 + 2/p - \delta$ and choose $N > 0$ large enough so that $s < 1 + 2/p^* - \delta$. From

(3.78), we apply the Bernstein inequalities again and the fact that $|k-j| \leq 4$ to get

$$\begin{aligned}
&\|T_{m_B}(\triangle_\ell \partial_i \widetilde{f}, \triangle_k \widetilde{g})\|_{L^p} \qquad (3.79)\\
&\quad\lesssim 2^{-(s+t)k} 2^{(2/p)k} \|\widetilde{g}\|_{\dot{B}^t_{p,\infty}} \sum_{\ell \leq k-3} 2^{-(1+2/p^*-s)(k-\ell)} 2^{s\ell} \|\triangle_\ell \widetilde{f}\|_{L^p}\\
&\quad\lesssim 2^{-(s+t-2/p)j} C_j \|\widetilde{f}\|_{\dot{B}^s_{p,q}} \|\widetilde{g}\|_{\dot{B}^t_{p,\infty}},
\end{aligned}$$

where

$$C_j := \sum_{j \geq \ell-2} 2^{-(1+2/p^*-s)(j-\ell)} 2^{s\ell} \|\triangle_\ell \widetilde{f}\|_{L^p},$$

which satisfies $(C_j)_{j\in\mathbb{Z}} \in \ell^q$ since $s < 1 + 2/p^*$.

Similarly, since $s < 1 + 2/p^* - \delta$, from (3.77) we can estimate

$$\begin{aligned}
&\|T_{m_A}(\triangle_\ell \partial_i \widetilde{f}, \triangle_k \widetilde{g})\|_{L^p} \qquad (3.80)\\
&\lesssim \gamma^{1-\delta/\alpha} 2^{-(\delta-\alpha+s+t-2/p)j} \|\widetilde{g}\|_{\dot{B}^t_{p,\infty}} \sum_{\ell \leq k-3} 2^{-(1+2/p^*-\delta-s)(k-\ell)} 2^{s\ell} \|\triangle_\ell \widetilde{f}\|_{L^p}\\
&\lesssim \gamma^{(\alpha-\delta)/\alpha} 2^{(\alpha-\delta)j} 2^{-(s+t-2/p)j} C_j \|\widetilde{f}\|_{\dot{B}^s_{p,q}} \|\widetilde{g}\|_{\dot{B}^t_{p,\infty}},
\end{aligned}$$

where

$$C_j := \sum_{j \geq \ell-2} 2^{-(1+2/p^*-\delta-s)(j-\ell)} 2^{s\ell} \|\triangle_\ell \widetilde{f}\|_{L^p},$$

which satisfies $(C_j)_{j\in\mathbb{Z}} \in \ell^q$ since $s < 1 + 2/p^* - \delta$.

Combining the estimates (3.64), (3.62), (3.63), (3.79), and (3.80) completes the proof of Theorem 32.

3.4 PROOF OF MAIN THEOREM

The proof will proceed in three steps. In the first step we will make two preliminary estimates. Next, we will establish properties for the approximating sequence. Finally, we conclude the proof by making the relevant a priori estimates.

3.4.1 PART I: PRELIMINARY ESTIMATES

We will need to control the linear term that appears from differentiating with respect to t, a term of the form $G_{\lambda t^{\alpha/\kappa}} f$ in the a priori estimates. To do so, we adapt the approach in [70] where the L^2 case is dealt with, and modify the proof to accomodate the general case of $p \neq 2$.

Lemma 43. Let $0 < \alpha < \kappa$ and $1 \leq p \leq \infty$. If $\Lambda^\alpha f, G_\gamma \Lambda^\kappa f \in L^p$, then

$$\|G_\gamma \Lambda^\alpha \triangle_j f\|_{L^p} \lesssim \|\Lambda^\alpha \triangle_j f\|_{L^p} + \gamma^{-(1-\kappa/\alpha)} \|G_\gamma \Lambda^\kappa \triangle_j f\|_{L^p}, \tag{3.81}$$

for all $j \in \mathbb{Z}$.

Proof. Fix an integer k, to be chosen later, such that $N := 2^{k-3}$. Denote by $\breve{\triangle}_j$ the augmented operator $\triangle_{j-1} + \triangle_j + \triangle_{j+1}$. Observe that

$$G_\gamma \Lambda^\alpha \triangle_j f = G_\gamma S_k (\Lambda^\alpha \triangle_j f) + \Lambda^{-(\kappa-\alpha)} (I - S_k) \triangle_j (G_\gamma \Lambda^\kappa \breve{\triangle}_j f).$$

Observe that $G_\gamma S_k \in L^1$. Indeed, by Lemma 26 we have

$$\|G_\gamma S_k\|_{L^1} \leq \sum_{n=0}^{\infty} \frac{\lambda^n \gamma^n}{n!} \|\Lambda^{\alpha n} S_k\|_{L^1} \leq e^{c\gamma 2^{k\alpha}}, \tag{3.82}$$

for some absolute constant $c > 0$. On the other hand, observe that $\check{m} := \Lambda^{-(\kappa-\alpha)}(I - S_k)\triangle_j$ is smooth with compact support. Let $g := G_\gamma \Lambda^\kappa \breve{\triangle}_j f$. We consider three cases.

If $2^{j+2} \leq N$, then $g \equiv 0$. If $N \leq 2^{j-2}$, then Lemma 26 and Young's convolution inequality implies that

$$\|T_m g\|_{L^1} \lesssim 2^{-(\kappa-\alpha)j} \lesssim N^{-(\kappa-\alpha)},$$

where T_m is convolution with $\check{m}$. Similarly, if $2^{j-1} \leq N \leq 2^{j+1}$, then

$$\|T_m g\|_{L^1} \lesssim N^{-(\kappa-\alpha)}. \tag{3.83}$$

Therefore, for any $N > 0$

$$\|G_\gamma \Lambda^\alpha \triangle_j f\|_{L^p} \lesssim e^{\gamma N^\alpha} \|\Lambda^\alpha \triangle_j f\|_{L^p} + N^{-(\kappa-\alpha)} \|G_\gamma \Lambda^\kappa \breve{\triangle}_j f\|_{L^p}.$$

Finally, choose $k := [\alpha^{-1} \log_2(1/\gamma)]$, where $[x]$ denotes the greatest integer $\geq x$. Then $N \sim \gamma^{-1/\alpha}$, which gives (3.81). $\square$

We will also require the following properties for the solution to the linear heat equation (3.8).

Lemma 44. Let $\alpha < \kappa$, $\sigma > 0$, and $\beta \geq 0$ and suppose that $\theta_0 \in \dot{B}^\sigma_{p,q}(\mathbb{R}^2)$. Then

(i) $\|e^{-(\cdot)\Lambda^\kappa} \theta_0\|_{X_T} \lesssim \|\theta_0\|_{\dot{B}^\sigma_{p,q}}$, for any $T \geq 0$, and

(ii) $\lim_{T\to 0} \|e^{-(\cdot)\Lambda^\kappa} \theta_0\|_{X_T} = 0$.

Proof. Observe that for $b < 1$, we have $e^{ax^b - cx} \leq 1$ for $x > 1$ and $e^{ax^b - cx} \lesssim e^{-cx}$ for $0 \leq x \leq 1$. If $t2^{j\kappa} \leq 1$, then arguing as in Lemma 43

$$\|e^{\lambda t^{\alpha/\kappa} \Lambda^\alpha} e^{-t\Lambda^\kappa} \triangle_j \theta_0\|_{L^p} \lesssim e^{c_1 \lambda (t 2^{j\kappa})^{\alpha/\kappa}} \|e^{-t\Lambda^\kappa} \triangle_j \theta_0\|_{L^p} \lesssim e^\lambda \|e^{-t\Lambda^\kappa} \triangle_j \theta_0\|_{L^p},$$

for some $c_1 > 0$. If $t2^{j\kappa} > 1$, then arguing as in Lemma 43 and applying Lemma 29

$$\|e^{\lambda t^{\alpha/\kappa}\Lambda^\alpha} e^{-t\Lambda^\kappa} \triangle_j \theta_0\|_{L^p} \lesssim e^{c_1\lambda t^{\alpha/\kappa} 2^{j\alpha} - c_2 t 2^{j\kappa}} \|e^{-(t/2)\Lambda^\kappa} \triangle_j \theta_0\|_{L^p} \lesssim \|e^{-c_3 t\Lambda^\kappa} \triangle_j \theta_0\|_{L^p}.$$

for some $c_1, c_2, c_3 > 0$. Therefore, a final application of Lemma 29 proves

$$\|e^{\lambda t^{\alpha/\kappa}\Lambda^\alpha} e^{-t\Lambda^\kappa} \triangle_j \theta_0\|_{L^p} \lesssim \|e^{-c_3 t\Lambda^\kappa} \triangle_j \theta_0\|_{L^p} \lesssim e^{-c_4 t 2^{j\kappa}} \|\triangle_j \theta_0\|_{L^p}, \tag{3.84}$$

for some $c_4 > 0$. Now by (3.84) we have

$$\begin{aligned} \|e^{\lambda t^{\alpha/\kappa}\Lambda^\alpha} e^{-t\Lambda^\kappa} \theta_0\|^q_{\dot{B}^{\sigma+\beta}_{p,q}} &= \sum_j 2^{(\sigma+\beta)jq} \|e^{\lambda t^{\alpha/\kappa}\Lambda^\alpha - t\Lambda^\kappa} \triangle_j \theta_0\|^q_{L^p} \\ &\lesssim \sum_j 2^{\beta jq} e^{-qc_4 t 2^{j\kappa}} \left(2^{\sigma j} \|\triangle_j \theta_0\|_{L^p}\right)^q \\ &\lesssim t^{-(\beta q)/\kappa} \|\theta_0\|^q_{\dot{B}^\sigma_{p,q}}. \end{aligned} \tag{3.85}$$

This proves (i). Now we prove (ii). Then for let $\epsilon > 0$, there exists $\theta_0^\epsilon \in \mathcal{S}$ such that $\mathscr{F}\theta_0^\epsilon$ is supported away from the origin and $\|\theta_0 - \theta_0^\epsilon\|_{\dot{B}^\sigma_{p,q}} < \epsilon$. In particular, $\theta_0^\epsilon \in \dot{B}^{\sigma+\beta}_{p,q}$. Observe that for $0 < t \leq T$

$$\begin{aligned} \|e^{-t\Lambda^\kappa} \widetilde{\theta}_0\|_{\dot{B}^{\sigma+\beta}_{p,q}} &\lesssim \|e^{-t\Lambda^\kappa} \widetilde{\theta}_0^\epsilon\|_{\dot{B}^{\sigma+\beta}_{p,q}} + \|e^{-t\Lambda^\kappa} \widetilde{\theta}_0 - e^{-t\Lambda^\kappa} \widetilde{\theta}_0^\epsilon\|_{\dot{B}^{\sigma+\beta}_{p,q}} \\ &\lesssim \|\theta_0^\epsilon\|_{\dot{B}^{\sigma+\beta}_{p,q}} + \|e^{\lambda t^{\alpha/\kappa}\Lambda^\alpha} e^{-t\Lambda^\kappa} (\theta_0 - \theta_0^\epsilon)\|_{\dot{B}^{\sigma+\beta}_{p,q}} \\ &\lesssim t^{-\beta/\kappa} \left(T^{\beta/\kappa} \|\theta_0^\epsilon\|_{\dot{B}^{\sigma+\beta}_{p,q}}\right) + t^{-\beta/\kappa} \|\theta_0 - \theta_0^\epsilon\|_{\dot{B}^\sigma_{p,q}}, \end{aligned}$$

where we have applied (3.85) to $\theta_0 - \theta_0^\epsilon$. This implies (ii) and we are done. □

3.4.2 PART II: APPROXIMATING SEQUENCE

Now let us consider the sequence of approximate solutions θ^n determined by

$$\begin{cases} \partial_t \theta^{n+1} + \Lambda^\kappa \theta^{n+1} + u^n \cdot \nabla \theta^{n+1} = 0 \text{ in } \mathbb{R}^2 \times \mathbb{R}_+, \\ u^n = (-R_2 \theta^{n+1}, R_1, \theta^n) \text{ in } \mathbb{R}^2 \times \mathbb{R}_+, \\ \theta^{n+1}\big|_{t=0} = \theta_0 \text{ in } \mathbb{R}^2, \end{cases} \tag{3.86}$$

for $n = 1, 2, \ldots$, and where θ^0 satisfies the heat equation

$$\begin{cases} \partial_t \theta^0 + \Lambda^\kappa \theta^0 = 0 \text{ in } \mathbb{R}^2 \times \mathbb{R}, \\ \theta^0\big|_{t=0} = \theta_0 \text{ in } \mathbb{R}^2. \end{cases} \tag{3.87}$$

It is well-known that θ^n is Gevrey regular for $n \geq 0$. In particular, we may define

$$\widetilde{\theta}^n(s) := G_\gamma \theta^n, \text{ and } \widetilde{u}^n(s) := G_\gamma u^n(s), \tag{3.88}$$

where we choose $\gamma = \gamma(s) := \lambda s^{\alpha/\kappa}$. It is shown in [14] that there exists a subsequence of $(\theta^n)_{n \geq 0}$ that converges in $L^p_{loc}(\mathbb{R}^+ \times \mathbb{R}^2)$ to some function $\theta \in C([0,T); \dot{B}^\sigma_{p,q})$, where $\sigma := 1 + 2/p - \kappa$, and which satisfies (3.1) in the sense of distribution, provided that either T or $\|\theta_0\|_{\dot{B}^\sigma_{p,q}}$ is sufficiently small. Additionally, we will show that the approximating sequence satisifies

$$\sup_{0<t<T} t^{\beta/\kappa} \|\theta^n(t)\|_{\dot{B}^{\sigma+\beta}_{p,q}} \lesssim \|\theta_0\|_{\dot{B}^\sigma_{p,q}} \text{ and } \lim_{T \to 0} \sup_{0<t<T} t^{\beta/\kappa} \|\theta^n(t)\|_{\dot{B}^{\sigma+\beta}_{p,q}} = 0, \tag{3.89}$$

for any $0 < \beta < \kappa/2$ and $n \geq 0$, where the suppressed constant above is independent of n. Whenceforth, to prove Theorem 30 it will suffice to obtain *a priori* bounds for $\|\theta^n(\,\cdot\,)\|_{X_T}$, independent of n.

To prove (3.89), we follow [68]. First observe that $\theta^0 = e^{-t\Lambda^\kappa}\theta_0$. Then by Lemma 44 we have

$$t^{\beta/\kappa}\|\theta^0\|_{\dot{B}^{\sigma+\beta}_{p,q}} \lesssim \|\theta_0\|_{\dot{B}^{\sigma}_{p,q}} \text{ and } \lim_{T\to 0}\sup_{0<t<T} t^{\beta/\kappa}\|e^{-t\Lambda^\kappa}\theta^0\|_{\dot{B}^{\sigma+\beta}_{p,q}} = 0.$$

We proceed by induction. Assume that (3.89) holds for some $n > 0$.

We apply $\triangle_j$ to (3.86) to obtain

$$\partial_t\theta_j^{n+1} + \Lambda^\kappa\theta_j^{n+1} + \triangle_j(u^n\cdot\nabla\theta^{n+1}) = 0. \tag{3.90}$$

Then we take the L^2 inner product of (3.91) with $|\theta_j|^{p-2}\theta_j$ and use the fact that $\nabla\cdot u^n = 0$ to write

$$\frac{1}{p}\frac{d}{dt}\|\theta_j^{n+1}\|_{L^p}^p + \int_{\mathbb{R}^2}\Lambda^\kappa\theta_j^{n+1}|\theta_j^{n+1}|^{p-2}\theta_j^{n+1}\,dx = -\int_{\mathbb{R}^2}[\triangle_j, u^n]\nabla\theta_j^{n+1}|\theta_j^{n+1}|^{p-2}\theta_j^{n+1}\,dx. \tag{3.91}$$

Note that we used the fact that

$$\int_{\mathbb{R}^2} u^n\cdot\nabla\theta_j^{n+1}|\theta_j|^{p-2}\widetilde{\theta}_j^{n+1}\,dx = 0, \tag{3.92}$$

which one obtains by integrating by parts and invoking the fact that $\nabla\cdot u^n = 0$ for all $n > 0$. Now, we apply Lemma 28, Lemma 27, and Hölder's inequality, so that after dividing by $\|\theta_j\|_{L^p}^{p-1}$, (3.91) becomes

$$\frac{d}{dt}\|\theta_j^{n+1}\|_{L^p} + C2^{\kappa j}\|\theta_j^{n+1}\|_{L^p} \lesssim \|[\triangle_j, u^n]\nabla\theta^{n+1}\|_{L^p}.$$

Let $\beta < \kappa/2$. By Corollary 33 with $s = \sigma + \beta$ and $t = 2/p - \kappa + \beta$ we get

$$\frac{d}{dt}\|\theta_j^{n+1}\|_{L^p} + C2^{\kappa j}\|\theta_j^{n+1}\|_{L^p} \lesssim 2^{-((\sigma+\beta)-(\kappa-\beta))j} c_j \|\theta^n\|_{\dot{B}^{\sigma+\beta}_{p,q}} \|\theta^{n+1}\|_{\dot{B}^{\sigma+\beta}_{p,q}}.$$

Note that we have used boundedness of the Riesz transform. Thus, multiplying by $2^{(\sigma+\beta)j}$, then applying Gronwall's inequality gives

$$\|\theta^{n+1}(t)\|_{\dot{B}^{\sigma+\beta}_{p,q}} \lesssim \left(\sum_j \left(e^{-C2^{\kappa j}t} 2^{(\sigma+\beta)j} \|\triangle_j \theta_0\|_{L^p}\right)^q\right)^{1/q}$$
$$+ \left(\int_0^t \sum_j \left(e^{-C2^{\kappa j}(t-s)} 2^{(\kappa-\beta)j} c_j \|\theta^n(s)\|_{\dot{B}^{\sigma+\beta}_{p,q}} \|\theta^{n+1}(s)\|_{\dot{B}^{\sigma+\beta}_{p,q}} \, ds\right)^q\right)^{1/q}.$$

In particular, this implies

$$\begin{aligned} t^{\beta/\kappa}\|\theta^{n+1}(t)\|_{\dot{B}^{\sigma+\beta}_{p,q}} \lesssim t^{\beta/\kappa} &\left(\sum_j \left(e^{-C2^{\kappa j}t} 2^{(\sigma+\beta)j} \|\triangle_j \theta_0\|_{L^p}\right)^q\right)^{1/q} \\ &+ t^{\beta/\kappa} \left(\int_0^t s^{-2\beta/\kappa}(t-s)^{-(1-\beta/\kappa)} ds\right) \left(\sum_j c_j^q\right)^{1/q} \\ &\times \left(\sup_{0<t<T} t^{\beta/\kappa}\|\theta^n(s)\|_{\dot{B}^{\sigma+\beta}_{p,q}}\right) \left(\sup_{0<t<T} t^{\beta/\kappa}\|\theta^{n+1}(s)\|_{\dot{B}^{\sigma+\beta}_{p,q}}\right), \end{aligned} \tag{3.93}$$

where we have used the fact that

$$x^b e^{-ax^c} \lesssim a^{-b/c}. \tag{3.94}$$

Since $\beta < \kappa/2$, $(c_j)_{j\in\mathbb{Z}} \in \ell^q$ and

$$\int_0^t \frac{1}{s^{2\beta/\kappa}(t-s)^{1-\beta/\kappa}} \, ds \lesssim t^{-\beta/\kappa},$$

we actually have

$$\sup_{0<t<T} t^{\beta/\kappa}\|\theta^{n+1}(t)\|_{\dot{B}^{\sigma+\beta}_{p,q}} \lesssim \sup_{0<t<T} t^{\beta/\kappa}\left(\sum_j \left(e^{-C2^{\kappa j}t}2^{(\sigma+\beta)j}\|\triangle_j\theta_0\|_{L^p}\right)^q\right)^{1/q} + \left(\sup_{0<t<T} t^{\beta/\kappa}\|\theta^{n}(t)\|_{\dot{B}^{\sigma+\beta}_{p,q}}\right)\left(\sup_{0<t<T} t^{\beta/\kappa}\|\theta^{n+1}(t)\|_{\dot{B}^{\sigma+\beta}_{p,q}}\right), \tag{3.95}$$

In fact, (3.94) also implies

$$M(t) := t^{\beta/\kappa}\left(\sum_j \left(e^{-C2^{\kappa j}t}2^{(\sigma+\beta)j}\|\triangle_j\theta_0\|_{L^p}\right)^q\right)^{1/q} \lesssim \|\theta_0\|_{\dot{B}^{\sigma}_{p,q}}. \tag{3.96}$$

From Lemma 44 we know that

$$e^{-C2^{\kappa j}t}\|\triangle_j\theta_0\|_{L^p} \lesssim \|e^{-c' t\Lambda^\kappa}\triangle_j\theta_0\|_{L^p},$$

for some $c' > 0$, where $v_j = e^{-c't\Lambda^\kappa}\triangle_j\theta_0$ solves the heat equation

$$\begin{cases} \partial_t v + c'\Lambda^\kappa v = 0 \\ v(x,0) = \triangle_j\theta_0(x). \end{cases}$$

Hence

$$M(t) \lesssim \sup_{0<t<T} t^{\beta/\kappa}\|e^{-c't\Lambda^\kappa}\theta_0\|_{\dot{B}^{\sigma+\beta}_{p,q}},$$

so that arguing as in Lemma 44, we may deduce that

$$\lim_{T\to 0}\sup_{0<t<T} M(t) = 0. \tag{3.97}$$

Recall that by hypothesis, we have

$$\lim_{T\to 0}\sup_{0<t<T} t^{\beta/\kappa}\|\theta^n(t)\|_{\dot{B}^{\sigma+\beta}_{p,q}} = 0.$$

Then returning to (3.95), by hypothesis, we may choose T sufficiently small so that

$$\sup_{0<t<T} t^{\beta/\kappa}\|\theta^n(t)\|_{\dot{B}^{1+2/p-\kappa+\beta}_{p,q}} < 1/2.$$

This implies that

$$\sup_{0<t<T} t^{\beta/\kappa}\|\theta^{n+1}(t)\|_{\dot{B}^{\sigma+\beta}_{p,q}} \lesssim \sup_{0<t<T} M(t).$$

Finally, letting $T \to 0$ and invoking (3.97) completes the induction.

3.4.3 PART III: A PRIORI BOUNDS

Now we will demonstrate *a priori* bounds for $\|\theta^n(\,\cdot\,)\|_{X_T}$, independent of n. First apply $G_\gamma \triangle_j$ to (3.86). Using the fact that $G_\gamma, \triangle_j, \nabla$ are Fourier multipliers (and hence, commute), we obtain

$$\partial_t \widetilde{\theta}^{n+1}_j + \Lambda^\kappa \widetilde{\theta}^{n+1}_j + G_\gamma \triangle_j (u^n \cdot \nabla \theta^{n+1}) = \lambda^{\kappa/\alpha}\gamma^{1-\kappa/\alpha}\Lambda^\alpha \widetilde{\theta}^{n+1}_j, \tag{3.98}$$

where we have used the fact that $\gamma := \lambda t^{\alpha/\kappa}$. Now apply Lemma 28, Lemma 27, and Hölder's inequality, as well as Lemma 43 to obtain

$$\begin{aligned}&\frac{d}{dt}\|\widetilde{\theta}^{n+1}_j\|_{L^p} + C2^{\kappa j}\|\widetilde{\theta}^{n+1}_j\|_{L^p}\\ &\quad\lesssim \lambda^{\kappa/\alpha}\gamma^{1-\kappa/\alpha}\|\Lambda^\alpha \theta^{n+1}_j\|_{L^p} + \lambda^{\kappa/\alpha}\|\Lambda^\kappa \widetilde{\theta}^{n+1}_j\|_{L^p} + \|[G_\gamma \triangle_j, u^n]\nabla\theta^{n+1}\|_{L^p}.\end{aligned}$$

We choose $\lambda > 0$ small enough so that Lemma 26 implies

$$\frac{d}{dt}\|\widetilde{\theta}_j^{n+1}\|_{L^p} + C2^{\kappa j}\|\widetilde{\theta}_j^{n+1}\|_{L^p} \lesssim \gamma^{1-\kappa/\alpha}2^{\alpha j}\|\theta_j^{n+1}\|_{L^p} + \|[G_\gamma\triangle_j, u^n]\nabla\theta^{n+1}\|_{L^p}. \tag{3.99}$$

Now choose $\alpha < \kappa$, $0 < \beta < \min\{\alpha, \kappa/2\}$ and $\delta > 0$ such that

$$\alpha < \delta + \beta < \kappa < \frac{1}{2} + \frac{1}{p} + \beta. \tag{3.100}$$

Then by Theorem 32 with $s = \sigma + \beta$ and $t = 2/p - \kappa + \beta$, we have

$$\begin{aligned}\frac{d}{dt}\|\widetilde{\theta}_j^{n+1}\|_{L^p} &+ 2^{\kappa j}\|\widetilde{\theta}_j^{n+1}\|_{L^p} \\ &\lesssim \gamma^{1-\kappa/\alpha}2^{\alpha j}\|\theta_j^{n+1}\|_{L^p} \\ &\quad + 2^{-((\sigma+\beta)-(\kappa-\beta))j}C_j\gamma^{(\alpha-\delta)/\alpha}2^{(\alpha-\delta)j}\|\widetilde{\theta}^n\|_{\dot{B}^{\sigma+\beta}_{p,q}}\|\widetilde{\theta}^{n+1}\|_{\dot{B}^{\sigma+\beta}_{p,q}} \\ &\quad + 2^{-((\sigma+\beta)-(\kappa-\beta))j}C_j\|\widetilde{\theta}^n\|_{\dot{B}^{\sigma+\beta}_{p,q}}\|\widetilde{\theta}^{n+1}\|_{\dot{B}^{\sigma+\beta}_{p,q}},\end{aligned}$$

Now by Gronwall's inequality, for $t \geq 0$ we have

$$\begin{aligned}2^{(\sigma+\beta)j}\|\widetilde{\theta}_j^{n+1}(t)\|_{L^p} &\lesssim 2^{\beta j}e^{-C2^{\kappa j}t}2^{\sigma j}\|\triangle_j\theta_0\|_{L^p} \\ &+ \int_0^t \gamma(s)^{1-\kappa/\alpha}2^{\alpha j}e^{-C(t-s)2^{\kappa j}}2^{(\sigma+\beta)j}\|\theta_j^{n+1}(s)\|_{L^p}\, ds \\ &+ C_j\int_0^t \gamma(s)^{(\alpha-\delta)/\alpha}2^{(\alpha-\delta+\kappa-\beta)j}e^{-C(t-s)2^{\kappa j}}\|\widetilde{\theta}^n(s)\|_{\dot{B}^{\sigma+\beta}_{p,q}}\|\widetilde{\theta}^{n+1}(s)\|_{\dot{B}^{\sigma+\beta}_{p,q}}\, ds \\ &+ C_j\int_0^t 2^{(\kappa-\beta)j}e^{-C(t-s)2^{\kappa j}}\|\widetilde{\theta}^n(s)\|_{\dot{B}^{\sigma+\beta}_{p,q}}\|\widetilde{\theta}^{n+1}(s)\|_{\dot{B}^{\sigma+\beta}_{p,q}}.\end{aligned}$$

Substituting $\gamma(s) = \lambda s^{\alpha/\kappa}$, applying the decay properties of the heat kernel $e^{-C(t-s)2^{\kappa j}}$, Minkowski's inequality, and by definition of the space X_T, we arrive at

$$\begin{aligned}\|\widetilde{\theta}^{n+1}(t)\|_{\dot{B}^{\sigma+\beta}_{p,q}} \lesssim & t^{-\beta/\kappa}\|\theta_0\|_{\dot{B}^{\sigma}_{p,q}} \\ & + \left(\int_0^t s^{-(1-(\alpha-\beta)/\kappa)}(t-s)^{-\alpha/\kappa}\, ds\right)\left(\sup_{0<t\leq T} t^{\beta/\kappa}\|\theta^{n+1}(t)\|_{\dot{B}^{\sigma+\beta}_{p,q}}\right) \\ & + \left(\int_0^t s^{(\alpha-\delta-2\beta)/\kappa}(t-s)^{-(\alpha-\delta+\kappa-\beta)/\kappa}\, ds\right)\|\theta^n\|_{X_T}\|\theta^{n+1}\|_{X_T} \\ & + \left(\int_0^t s^{-2\beta/\kappa}(t-s)^{-(\kappa-\beta)/\kappa}\, ds\right)\|\theta^n\|_{X_T}\|\theta^{n+1}\|_{X_T}\end{aligned}$$

Since $\beta < \min\{\alpha, \kappa/2\}$, $\alpha < \beta + \delta$, and $\alpha < \kappa$, we deduce after an application of (3.89) that

$$\|\theta^{n+1}\|_{X_T} \leq C_1\|\theta_0\|_{\dot{B}^{\sigma}_{p,q}} + C_2\|\theta^n\|_{X_T}\|\theta^{n+1}\|_{X_T}, \tag{3.101}$$

for some constants $C_1, C_2 > 1$. By Lemma 44 we have

$$\|\theta^0\|_{X_T} \leq C_3\|\theta_0\|_{\dot{B}^{\sigma}_{p,q}} \leq 2(C_1 \vee C_3)\|\theta_0\|_{\dot{B}^{\sigma}_{p,q}}, \tag{3.102}$$

for some constant $C_3 > 1$. Let $C_4 := 2(C_1 \vee C_3)$ and assume that $\|\theta_0\|_{\dot{B}^{\sigma}_{p,q}} \leq (2C_2C_4)^{-1}$. If $\|\theta^n\|_{X_T} \leq C_4\|\theta_0\|_{\dot{B}^{\sigma}_{p,q}}$ for some $n > 0$, then from (3.101), we get

$$\frac{1}{2}\|\theta^{n+1}\|_{X_T} \leq C_1\|\theta_0\|_{\dot{B}^{\sigma}_{p,q}}. \tag{3.103}$$

Therefore, by induction $\|\theta^n\|_{X_T} \leq C_4\|\theta_0\|_{\dot{B}^{\sigma}_{p,q}}$ for all $n \geq 0$.

For arbitrary $\theta_0 \in \dot{B}^{\sigma}_{p,q}$, we can deduce uniform bounds for $\{\theta^n\}_{n\geq 0}$ by induction similarly. To this end, we first observe that by Lemma 44, there exists $T_1 > 0$ such that $\|\theta^0\|_{X_{T_1}} \leq C$, where $C < (2C_2)^{-1}$. We can also choose $T_0 = T_0(\theta_0)$ such that $\sup_{0<t<T_0} M(t) \leq C(2C_1)^{-1}$, where $M(t)$ is defined as in (3.96). Now let $T^* := T \wedge T_0$. It

follows that $\|\theta^0\|_{X_{T^*}} \leq C$.

For $n > 0$, observe that similar to (3.101), we also have the estimate

$$\|\theta^{n+1}\|_{X_{T^*}} \leq C_1 \left(\sup_{0<t<T^*} M(t) \right) + C_2 \|\theta^n\|_{X_{T^*}} \|\theta^{n+1}\|_{X_{T^*}}. \tag{3.104}$$

If $\|\theta^k\|_{X_{T^*}} \leq C$, for all $0 < k \leq n$, then applying this to (3.104) and using the fact that $C < (2C_2)^{-1}$, we have

$$\|\theta^{n+1}\|_{X_{T^*}} \leq 2C_1 \left(\sup_{0<t<T^*} M(t) \right).$$

Since $\sup_{0<t<T^*} M(t) \leq C(2C_1)^{-1}$ we therefore have

$$\|\theta^{n+1}\|_{X_{T^*}} \leq C,$$

which completes the induction.

Remark 45. To replace X_T by Z_T as mentioned in Remark 1, one must first prove an analog of Lemma 44 (i) for the space Y_T to take care of the case $n = 0$. This follows easily from the proof of Lemma 44 by setting $\beta = 0$. Then for the case $n > 0$, one returns to (3.99) and applies Theorem 32 with $s = 1 + 2/p - \kappa + \beta$ and $t = 2/p - \kappa$, which forces the additional constraint $1/2 + 1/p + \beta/2 > \kappa$. One can then obtain uniform bounds on $\|\theta^n\|_{Y_T}$ by following steps similar to those made for estimating $\|\theta^n\|_{X_T}$, and taking advantage of the fact that $\|\theta^n\|_{X_T}$ is already uniformly bounded for all $n \geq 0$.

3.5 APPENDIX B

Lemma 46. Let $\alpha < 1$ and $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(\xi, \eta) := \|\xi\|^\alpha + \|\eta\|^\alpha - \|\xi + \eta\|^\alpha. \tag{3.105}$$

If $\|\xi\|/\|\eta\| \geq c$ for some $c > 0$, then there exists $\epsilon > 0$, depending only on c, such that $f(\xi, \eta) \geq \epsilon \|\eta\|^\alpha$.

Proof. Observe that

$$f(\xi, \eta) = \|\eta\|^\alpha \left(\left\| \frac{\xi}{\|\eta\|} \right\|^\alpha + 1 - \left\| \frac{\xi}{\|\eta\|} + \frac{\eta}{\|\eta\|} \right\|^\alpha \right).$$

Also observe that if R is a rotation matrix, then $f(R\xi, R\eta) = f(\xi, \eta)$. Thus, we may assume that $\|\xi\| \geq c$ and that $\eta = e_1$, where $e_1 := (1, 0)$. Now observe that

$$\begin{aligned} f(\xi, \eta) &= (\xi_1^2 + \xi_2^2)^{\alpha/2} + 1 - ((\xi_1 + \eta_1)^2 + (\xi_2 + \eta_2)^2)^{\alpha/2} \\ &= (\xi_1^2 + \xi_2^2)^{\alpha/2} + 1 - ((\xi_1 + 1)^2 + \xi_2^2)^{\alpha/2}. \end{aligned}$$

Let $x := \|\xi\|$. Then

$$f(\xi, \eta) = g_{\xi_1}(x) := x^\alpha + 1 - (x^2 + 1 + 2\xi_1)^{\alpha/2},$$

where $x \geq c$. Thus, we may assume $\xi_2 = 0$ and $|\xi_1| \geq c$. In particular, we may assume that $x = \xi_1$. Finally, elementary calculation shows that $g(x) := |x|^\alpha + 1 - |x + 1|^\alpha \geq \min\{g(-c), g(c)\} > 0$, provided that $|x| \geq c$. $\square$

Now we provide the proof of our multiplier theorem, Theorem 35.

Proof. By Proposition 47, we may assume that for each fixed $\xi \in \mathbb{R}^d$, $m(\xi, \eta)$ is supported in $[1/2 \le \|\eta\| \le 2] \subset [0,4]^d$ as a function of η. Thus, we may take the Fourier transform in the variables $\eta_1, \dots, \eta_d$, i.e.,

$$m(\xi, \eta) \sim \sum_{k \in \mathbb{Z}^d} \hat{m}_k(\xi) e^{ik\cdot\eta} \chi(\eta), \tag{3.106}$$

where $\hat{m}_k(\xi) := \hat{m}(\xi, k)$ is the k-th Fourier coefficient of m and $\chi(\eta) = 1$ for $1/2 \le \|\eta\| \le 2$ and is supported on $[1/4 \le \|\eta\| \le 4]$. In fact, we write $m(\xi, \eta)$ as

$$m(\xi, \eta) \sim \hat{m}_0(\xi)\chi(\eta) + \left(\sum_{k \in Z_0} + \dots + \sum_{k \in Z_{d-1}} \right) \hat{m}_k(\xi) e^{ik\cdot\eta} \chi(\eta), \tag{3.107}$$

where $Z_j \subset \mathbb{Z}^d$ is defined by

$$Z_j := \{k \in \mathbb{Z}^d : k_i = 0 \text{ for exactly } j \text{ many indices } i \text{ and } k_{i'} \neq 0 \text{ for } i' \neq i\}, \tag{3.108}$$

Observe that Z_j is precisely equal to $C(d, d-j)$ copies of $(\mathbb{Z} \setminus \{0\})^{d-j}$.

Using multi-index notation, observe that for each $k \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$, integration by parts gives

$$\hat{m}_k(\xi) = \int e^{-ik\cdot\eta} m(\xi, \eta) \; d\eta = c_\alpha (-ik)^{-\alpha} \widetilde{m}_{k,\alpha}(\xi),$$

for all $\alpha \in \mathbb{N}^d$, where

$$\widetilde{m}_{k,\alpha}(\xi) := \int e^{-ik\cdot\eta} \partial_\eta^\alpha (m(\xi, \eta)\chi(\eta)) \; d\eta.$$

By (3.29), it follows that $m_0(\xi)$ is a Hörmander-Mikhlin multiplier. On the other hand, (3.29) and and the fact that χ is supported in $[1/4 \leq \|\eta\| \leq 4]$ implies

$$\begin{aligned}\left|\partial_\xi^\beta \widetilde{m}_{k,\alpha}(\xi)\right| &\lesssim \sum_{\alpha_1+\alpha_2=\alpha} \int \left|\partial_\xi^\beta \partial_\eta^{\alpha_1} m(\xi,\eta) \partial_\eta^{\alpha_2}\chi(\eta)\right| \, d\eta \\ &\lesssim_{\beta,\alpha,d} \|\xi\|^{-|\beta|} \int_{[1/4\lesssim\|\eta\|\lesssim 4]} \|\eta\|^{-|\alpha_1|} \, d\eta \\ &\lesssim_{\beta,\alpha,d} \|\xi\|^{-|\beta|}. \end{aligned} \tag{3.109}$$

Thus $\widetilde{m}_{k,\alpha}$ is also a Hörmander-Mikhlin multiplier for all $k \in \mathbb{Z}^d$ and $\alpha \in \mathbb{N}^d$. Moreover, note that the suppressed constant in (3.109) is independent of k.

Now for each $j = 1, \ldots, d$, choose a multi-index $a_j \in Z_j \cap \mathbb{N}^d$ so that $\sum_{k \in Z_j} k^{-a_j} < \infty$. Finally, observe that

$$\begin{aligned} T_m(f,g) &= T_{m_0}(f) T_\chi g + \sum_{j=1}^d \sum_{k\in Z_j} T_{m_k}(f) T_{\chi_k}(g) \\ &= T_{m_0}(f) T_\chi(g) + \sum_{j=1}^d \sum_{k\in Z_j} c_{\alpha_j} k^{-a_j} (T_{\widetilde{m}_{k,a_j}} f)(T_\chi \tau_{-k} g), \end{aligned}$$

where $\chi_k(\eta) := \chi(\eta) e^{ik\cdot\eta}$ and T_{m_k}, T_{χ_k} denote linear multiplier operators with symbols m_k, χ_k, respectively. Therefore, by Minkowski's inequality, Hölder's inequality, and the Hörmander-Mikhlin multiplier theorem we have

$$\|T_m(f,g)\|_{L^r} \lesssim_a \|f\|_{L^p} \|\chi\|_{L^1} \|g\|_{L^q},$$

where we have used Young's convolution inequality and translation invariance of dx, and the suppressed constant depends on $\sup_j \left(\sum_{k\in Z_j} k^{-a_j}\right)$.

□

The next proposition shows that Marcinkiewicz multipliers are dilation invariant. Thus, we may (isotropically) rescale the support of m without penalty.

Proposition 47. Let $1/r = 1/p + 1/q$ and $T_m : L^p \times L^q \to L^r$ be a bounded bilinear multiplier operator whose multplier, m, satisfies $m \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$

$$\left|\partial_\xi^{\beta_1}\partial_\eta^{\beta_2} m(\xi,\eta)\right| \lesssim_{\beta,d} \|\xi\|^{-|\beta_1|}\|\eta\|^{-|\beta_2|}, \tag{3.110}$$

for all $\xi, \eta \in \mathbb{R}^d \setminus \{0\}$ and multi-indices $\beta_1, \beta_2 \in \mathbb{N}^d$. Then T_{m_λ} is also bounded with the same operator norm, where m_λ is given by

$$m_\lambda(\xi,\eta) := m(\lambda\xi, \lambda\eta).$$

Proof of claim. We first show that m_λ also satisifes (3.110). Observe that

$$\partial_\xi^{\beta_1}\partial_\eta^{\beta_2} m_\lambda(\xi,\eta) = \lambda^{|\beta_1|+|\beta_2|}(\partial_\xi^{\beta_1}\partial_\eta^{\beta_2} m)(\lambda\xi,\lambda\eta).$$

Then since m satisifes (3.110) we have

$$\left|\partial_\xi^{\beta_1}\partial_\eta^{\beta_2} m_\lambda(\xi,\eta)\right| \lesssim \lambda^{|\beta_1|+|\beta_2|}\|\lambda\xi\|^{-|\beta_1|}\|\lambda\eta\|^{-|\beta_2|}.$$

Now we prove the claim. Indeed, let $f \in L^p, g \in L^q$, and $\lambda > 0$. Then

$$\begin{aligned}
T_{m_\lambda}(f,g)(x) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{ix\cdot(\xi+\eta)} m_\lambda(\xi,\eta)\hat{f}(\xi)\hat{g}(\eta)\ d\xi\ d\eta \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{ix\cdot(\xi+\eta)} m(\lambda\xi,\lambda\eta)\hat{f}(\xi)\hat{g}(\eta)\ d\xi\ d\eta \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i(x/\lambda)\cdot(\xi'+\eta')} m(\xi',\eta')\lambda^{-d}\hat{f}(\xi'/\lambda)\lambda^{-d}\hat{g}(\eta'/\lambda)\ d\xi'\ d\eta' \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i(x/\lambda)\cdot(\xi+\eta)} m(\xi,\eta)\widehat{f_\lambda}(\xi)\widehat{g_\lambda}(\eta)\ d\xi\ d\eta \\
&= T_m(f_\lambda, g_\lambda)(x/\lambda) = (T_m(f_\lambda, g_\lambda))_{1/\lambda}(x).
\end{aligned}$$

This implies

$$\|T_{m_\lambda}(f,g)\|_{L^r} = \lambda^{d/r}\|T_m(f_\lambda, g_\lambda)\|_{L^r}$$

$$\lesssim \lambda^{d/r}\|f_\lambda\|_{L^p}\|g_\lambda\|_{L^q} = \lambda^{d/r}\lambda^{-d/p}\lambda^{-d/q}\|f\|_{L^p}\|g\|_{L^q}.$$

In particular, $\|T_{m_\lambda}\| \leq \|T_m\|$. On the other hand, one can similarly argue

$$\|T_m(f,g)\|_{L^r} = \lambda^{-d/r}\|T_{m_{1/\lambda}}(f_{1/\lambda}, g_{1/\lambda})\|_{L^r}$$

$$\lesssim \lambda^{-d/r}\|f_{1/\lambda}\|_{L^p}\|g_{1/\lambda}\|_{L^q} = \lambda^{-d/r}\lambda^{d/p}\lambda^{d/q}\|f\|_{L^p}\|g\|_{L^q}.$$

Therefore $\|T_m\| \leq \|T_{m_{1/\lambda}}\|$. This completes the proof.

$\square$

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Curriculum Vita

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Education:

- Ph.D., Pure Mathematics, Indiana University, 2014
  - Advisor: Michael S. Jolly
- B.A., Mathematics, The College of New Jersey, 2008
- Mathematics Advanced Study Semester (MASS), Penn State University, Fall 2007

Employment:

- Associate Instructor, Indiana University Bloomington Fall 2009 – Summer 2014.
  - M014: Basic Algebra, Fall 2009
  - M119: Brief Survey of Calculus 1, Summer 2011, '13, Fall 2013
  - T101: Math for Elementary Teachers, Fall 2012, '13
  - J110: Introductory Problem Solving, Summer 2014
  - J111: Intro to College Math I, Fall 2010
  - J112: Intro to College Math II, Spring 2010, Fall 2011
  - J113: Intro to Calculus with Applications, Spring 2011, '12, Fall 2012

Publications:

- A. Biswas, M.S. Jolly, V.R. Martinez, E.S. Titi,"Dissipation length scale estimates for turbulent flows - a Wiener algebra approach," *Journal of Nonlinear Science*, 24:441-471, 2014.
- A. Biswas, V.R. Martinez, P.S. Silva, "On Gevrey regularity of the supercritical SQG equation in critical Besov spaces," arXiv.1312.5755, 2013 (submitted).

Talks:

- Tulane University, Applied and Computational Math Seminar

  Invited talk March 28, 2014

- University of Maryland-Baltimore County, Workshop on Analysis of Nonlinear PDEs and Fluid Flows

  Contributed talk January 19-20, 2014

- SIAM Conference on Analysis of Partial Differential Equations, Orlando, FL

  Invited talk in MS20 December 7-10, 2013

- AMS Fall Southeastern Sectional Meeting, Louisville, KY

  Invited talk in Special Session on PDEs from Fluid Mechanics October 5-6, 2013

- Stanford Summer School and Workshop: Recent Advances in PDEs and Fluids

  Contributed talk August 5-18, 2013

- Indiana University Dissipative Systems Workshop

  Contributed talk February 8-10, 2013

- 9th AIMS Conference, Orlando, FL

  Invited talk in Special Session #30 July 1-5, 2012

- Texas A&M Workshop on Study of Turbulence in Physical Systems Through Complex Singularities and Determining Modes

  Contributed talk February 17-20, 2012

Awards:

- Indiana University-Bloomington
  - Glenn Schober Travel Award, Spring 2014
  - Rothrock Teaching Award, Spring 2012
  - Matias Ochoada Fellowship, Fall 2011
  - Graduate Scholars Fellowship, 2008-2009 Academic Year
- SIAM Student Travel Award, Fall 2013
- MASS, Best performance on analysis exam, Fall 2007
- Pi Mu Epsilon Honor Society New Jersey Theta Chapter, 2006